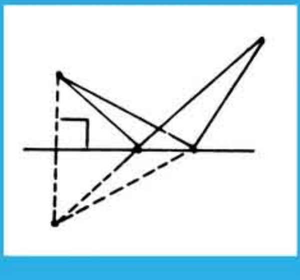
Mathematics in Commonwealth Schools

Report of a Specialist Conference
held at the University of the West Indies,
St. Augustine, Trinidad
September 1968



COMMONWEALTH SECRETARIAT

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Published by the
COMMONWEALTH SECRETARIAT

To be purchased from the
COMMONWEALTH SECRETARIAT
Printing Section
Marlborough House
London, S.W.1

S.B.N. 85092 000. 0.

Foreword

By W.H. Cockcroft

G.F. Grant, Professor of Mathematics, University of Hull,
and Chairman of the Conference

It is a pleasure to commend this report to all who are concerned with mathematical education. Two weeks is all too short a period of time for delegates from a variety of countries to be able to lift their eyes from their own pressing problems, to look at those of their fellow delegates, and then to come together in an understanding of common aims and objectives in their various educational programmes. It is a tribute to the educational ties which bind us in the Commonwealth, and to all the individual delegates, that in two weeks not only was there felt to be a common purpose in the work of the Conference, but also that the groundwork of this report could be laid so firmly.

Naturally most common ground was found in practice at the base of our educational pyramid, in primary education. This is reflected in the depth to which the Conference was able to go in considering work at this level. Secondary modes of education differ markedly from one part of the Commonwealth to another, but as the Secretary-General of the Commonwealth Secretariat has remarked elsewhere, the Commonwealth is not concentric, but polycentric; as the chapter on Commonwealth Co-operation in this report shows, we can draw strength from our wealth of differing patterns at secondary level rather than seeing weakening divisions in them.

It was not the purpose of the Conference to go beyond a setting out of basic principles: the prescription of administrative action was not in its brief. It was concerned to produce, just as other conferences of this kind have produced, a guide to action under the very different conditions which exist in our various countries. If the reader ever thinks the Conference was unaware of the hard facts of life in developing countries he should read the background papers submitted to the Conference by Governments. Delegates were only too well aware of the practical problems involved, in any country, if their suggested principles were to be acted on.

Not the least of the scarce resources, exacerbating the problems which exist in all our educational programmes, is the human one, the trained competent teacher. The need to help present and future teachers in today's climate of curriculum renewal and development was a matter of deep concern to delegates. This concern was reflected in their practical suggestions for the development of the principle of joint effort in introducing new material into the classroom, so that individual teachers can rely on the support and help of their peers in centres set aside for this purpose.

Such thinking reflected the conviction of all delegates that education is above all a practical human activity involving men and women, and children who rightly come into our schools in larger and larger numbers, in all their shapes and sizes, with their advantages and limitations, with both hopes and fears. If in dealing with mathematical educational principles this report reflects this broader conviction, then delegates will be well satisfied.

It was the hope of the Conference that the report would help those concerned to further mathematical education, at all levels, in all our countries. The efforts of the Conference were directed wholly towards that end. I know that delegates, authors and editors would wish the report to be judged by its influence in practice as much as, if not more than, for its intrinsic merit as a document about mathematical education.

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TEXT OF THE OPENING SPEECH BY DR. THE RT. HONOURABLE ERIC WILLIAMS, PRIME MINISTER OF TRINIDAD AND TOBAGO

It is a rare pleasure and a very real one that I have this morning – that of addressing this distinguished gathering on the occasion of the Opening of the Commonwealth Conference on Mathematics in Schools to which Trinidad and Tobago has the honour to be host. This honour is one that the Government and people of this country feel deeply, and on their behalf I wish to express now our sincere appreciation to the Commonwealth Secretariat, and to offer a warm welcome to all our visitors from the various countries of the Commonwealth. I trust that along with the benefits you will derive from your deliberations here, you will take back with you, after your too short stay among us pleasant memories of our land and our people.

The Commonwealth has in recent years, and for obvious reasons, been subject to a number of internal stresses and strains, but it is clear that whatever the tendencies to disunity and disharmony, and however strong may be the forces working in this direction, the machinery organised for Commonwealth co-operation in Education ever since the original Conference in Oxford in 1959 has served as a powerful link, binding the various member countries together, to their mutual advantage.

The Specialist Conferences which have been held over the years have been of particular interest and benefit to the developing countries, and if you wish to see monuments to the development taking place in this country I invite you to look around you.

For this University campus has grown and is growing out of the Imperial College of Tropical Agriculture, an Institution which served the Empire of yesteryear. Many of the countries represented here today have had some form of contact with us through the old Imperial College, but today we meet on a new level amid new buildings serving new purposes.

What was once a small college, imperial in scope but limited in the services it offered, has now become a spreading branch of the University of the West Indies offering courses in Agriculture, Engineering, Natural Sciences, Arts, Social Science, Management Studies, and Afro-Asian Studies. It has also sprouted an Institute of Education and an Institute of International Relations.

In this respect we are not unique. All of the countries of the Commonwealth, including those who are unfortunately not represented here today and including the older, developed countries are either achieving similar expansion of facilities for higher education or are having it thrust upon them by the force of circumstances.

These circumstances are basically very much the same in all countries, developed or developing, and the pressures differ rather in degree and intensity than in kind, for fundamentally the problems are those of shaping society to meet ever-changing needs. The solutions will be different from country to country, but the problems are the same. And it is clear that education has a central role to play.

While in all countries universities are expanding, new ones being created, other institutions of higher learning being incorporated into existing universities or being upgraded to university status in order to meet the demand for highly skilled manpower in all fields, the pressures from below to provide more and better facilities for primary and secondary education are, to put it mildly, at least as great.

From a quantitative point of view, the poorer countries suffer the greatest hardships, for education has to compete with other services for a share of the slender budgetary resources, and then the problems arise over how best to distribute this slender share among the different levels of education in order to effect a proper balance. An educated citizenry is necessary in order to make any economic development, and economic development is necessary in order to provide the facilities for education. This is the vicious circle in which so many countries find themselves trapped.

From the point of view of the content of the curriculum and teaching methods, this is particularly an area where different approaches and different solutions have to be sought according to the social, political, and economic circumstances of the individual countries. However, it is equally clear that a pooling and a sharing of experiences, interchanges of personnel and other forms of technical assistance can be of the utmost value in pointing a way towards possible solutions.

This is the justification for the Commonwealth Education Conferences and the Specialist Conferences organised by the Commonwealth Education Liaison Committee. These conferences must be more than expensive academic exercises. They must bear fruit. This is why the Education Division of the Commonwealth Secretariat is expanding in scope and in personnel as was agreed to at the recent Fourth Commonwealth Conference.

In particular, the use of local materials as teaching aids, the teaching of English either as a first or as a second language, the development of programmes in social studies, the role of the folk culture in the creative subjects in the curriculum are illustrations of individual problems which are more easily solved by an open sharing of ideas, experiences, and information. In this way most countries will be better able to avoid the pitfalls of expensive procedures of trial and error on the one hand, and of slavish imitation on the other.

At first glance, mathematics might appear to some to be a universal language cutting across all barriers of nationality, and independent of the stage of development of any particular country. Thus, it might be argued that the problems of mathematics and the teaching of mathematics in an advanced country are the same as those in an under-developed country. This is plausible, and clearly contains a large element of truth.

But if we go a little beneath the surface it becomes obvious that the cultural patterns of a community exert a determining force on mathematics in the schools – the content, the teaching methods, the depth of knowledge necessary or even desirable. And the political and economic realities likewise help to determine what is practicable.

In short, even a subject such as mathematics cannot be divorced from its social context, and so all the time that we are looking at mathematics in the schools we are also looking at the schools themselves, the education system and the whole complex of political, economic and social institutions that combine to form the culture of the community.

To illustrate this a little further, without wishing to labour a point that may seem obvious to you; it is impossible to have a meaningful discussion on mathematics teaching without paying particular attention to such considerations as the role and status of women in the society, the degree of industrialization or the general sophisitication of all levels of the community.

The history of the development of mathematics shows quite clearly the extent to which it is just one facet of cultural expression. However, you will be concerned in the next few weeks not with the past but with the present and more particularly I hope with the future.

The twenty-first century is bearing down hard upon us in the present-day world, while some countries have hardly begun to enter the twentieth. Time does not stand still, nor do the aspirations of the people, as those who serve in the political field know only too well.

I think all developing countries, government and people alike, realise that economic development depends largely on the development of human resources, hence the central role of education. Everywhere there has been a realisation of the importance of communication through all media, and the printed word remains one of the most powerful of these. Campaigns against illiteracy have been waged and are being waged on national and international scales in order to remove this barrier to progress.

The time has come for similar onslaughts to be made against innumeracy, which increasingly in the world today and inexorably in the world tomorrow is and will be as much a crime as illiteracy has been up to now.

I do not need to deal at length with the vocational aspects of education in mathematics. It is obvious that more and more people will need to use the tools of mathematics in the course of their employment. The growing need in all societies for technicians, engineers, economists, statisticians, experts in the various technologies, (to name a few) bears ample testimony to this. Gone are the days when virtually the only purpose of mathematics in the schools was to produce a steady supply of teachers of mathematics in the schools. It is small wonder that this self-perpetuating process of in-breeding, without a larger aim and purpose behind it, served to instil into most pupils the idea that mathematics was largely a form of magic used by the initiated to select the few for further initiation rites and to bamboozle the many.

The so-called New Mathematics and an enlightened approach generally to the content and purpose of mathematics will go far towards removing many absurdities and misconceptions from the minds of educators and fears and mystifications from the minds of pupils.

And this is not merely necessary; it is urgent. For quite apart from the need to train skilled manpower in the use of mathematical tools, it is becoming more and more obvious that science and mathematics form an essential part of the general education of all people, if they are to understand the world around them at all. We can no longer think of a sound general education if the mass of the population is disqualified from a simple appreciation of the mathematics of the economy (at the domestic and the governmental levels), of the statistical data with which they are continually bombarded, of the mathematical bases of most of the building construction and domestic equipment with which they are in daily contact.

Further, the idea is being demolished that mathematical ability and aptitude are gifts conferred by the Creator on a favoured few. Exceptional geniuses there always will be, who will come when they will come and where they will come, but there are many more fish in the sea than were once thought.

This conference has been called because there has been recognition of the points I have been trying to make, and of others, equally or more important, which I have not dared to touch upon for fear of revealing the extent of my own innumeracy.

Your programme is, to say the least, exciting, and all aspects of mathematics in schools are covered in a practical yet imaginative manner. I do not need to say that conferences of this sort are a beginning, not an end, for the work you will be doing over the next two weeks is but a spark that I hope will result in firing with enthusiasm all those who are concerned with education in general and with mathematics in particular.

Before I conclude, Mr. Chairman, I should like to repeat my words of welcome to our visitors from the other lands of the Commonwealth, and to stress the deep appreciation of the Government and people of Trinidad and Tobago of the honour of hosting this Conference, as well as my own personal appreciation of the honour you have done me in asking me to address you.

It now gives me the greatest pleasure to declare open this Conference on Mathematics in Schools and to wish that your deliberations will be both pleasant and fruitful.

TEXT OF THE CLOSING SPEECH BY SENATOR THE HON. DONALD PIERRE, MINISTER OF EDUCATION AND CULTURE, GOVERNMENT OF TRINIDAD AND TOBAGO

We have now come to the end of this two week conference which has been exhaustive as well as, I am afraid for many of you, exhausting. However, physical tiredness resulting from prolonged and intensive efforts is easily relieved by a short rest from labour, and I am sure that, however much you may be feeling now the strain of your efforts during the last few weeks, the process of recovery will be a short one even if you had to pick up again almost immediately the threads of your routine duties. On the other hand, I am also confident that the mental stimulation has been such that it will last a long time and will be reflected in the work you will be doing for years to come.

Your programme has been very full and has included every aspect of mathematics in both primary and secondary schools. It would be idle and perhaps dangerous to single out any one topic as more important than the others. But I hope you will pardon me for making special reference to the training of teachers since this seems to me to be the basic topic. It is the bedrock on which a sound superstructure can be erected, or else the quicksand on which it will flounder. For it is generally accepted that there is no greater fear of the child in school than the fear of mathematics. And this fear is not innate, nor is it inspired by itself. It is a fear which is instilled unconsciously and sometimes even, I regret to say, consciously by those who profess to teach mathematics.

I do not think I have overstated the case at least as far as the recent past is concerned for the ways of enthusiasm for the New Mathematics and for new approaches to the teaching of mathematics. Indeed the very fact of this conference suggests that there has been recognition of the need for new attitudes on the part of teachers. You will remember the central character in Molière's play who discovered with delight that he had been speaking prose all his life without realizing it.

Perhaps the first step to success in mathematics teaching in the schools will be shown when the children discover with equal delight that they have been doing mathematics all the time without realizing it. But, first, the teachers must themselves find delight in the subject. In fact, I venture to suggest that success depends finally on those two words – discovery and delight. The insight that the teacher has into the nature, and the purpose, and the structure of mathematics has to be transmitted to the children. In fact, if things continue as they have been going on recently, we may have to find a new word to replace the word teacher as the emphasis has shifted so much from teaching to creating situations in which learning best takes place.

I do not wish to dwell on details of the programme for I know that you will be leaving here with notes, with ideas and with good intentions. But let me remind you also that the road to hell is paved with good intentions.

The conference has served other purposes than the central one of providing a setting for these ideas. First, it has served to bring together people from all parts of the Commonwealth, who have worked and lived together harmoniously on this campus for two weeks. It is impossible to exaggerate the importance of this especially in a world where wars and rumours of wars are more than mere words.

Next, the personal friendships made here will, I hope survive over the years and over the miles which will soon be separating us again, just as the professional ideas and insights will survive.

Finally, and equally important, there has been a coming together, and a sharing of experiences of people at all levels in the education system, primary, secondary and intermediate. Too often in the past have these three levels gone their own sweet ways; too often in contrary directions. If this conference has done no more than to foster the recognition of the need for seeing mathematics whole, and seeing the education system whole, it would have done a tremendous amount of good.

This specialist conference, organized by the Commonwealth Secretariat has ended. I suspect that the Secretariat will begin planning the next one from this afternoon. Indeed, the success of the conference rests largely on the careful detailed planning that has gone into it. I congratulate the organisers and all those who have slaved to make it a reality. It has been an honour to my country to host this conference, and it has been a pleasure to have so many visitors from so many different lands. The weather has not been too unkind to you, and I hope that in the few hours in which you have been free from the rigors of a two-week exercise, you have been able to see something of our country and our people. We shall always retain pleasant memories of your stay here and I hope that your memories will be as pleasant.

In conclusion, Mr. Chairman, I should like to express thanks to you first, for your very affable manner which, no doubt, has played an important part in making this conference the success that it is. I also want to express thanks on behalf of the Government and people of Trinidad and Tobago to all the delegates here assembled, who have expressed warm feelings for our Government and our people, and particularly to the delegate from Africa. I will communicate to the Prime Minister his very warm tidings and greetings.

May I close, therefore, Mr. Chairman, by again expressing the gratitude of the Government of Trinidad and Tobago for hosting this conference. It was indeed a privilege and I do hope that in the very near future we shall be able to play host to other conferences. Once again, thank you, Ladies and Gentlemen, for spending this stay with us in Trinidad and Tobago for two weeks.

Mr. Chairman, do not believe that I was negligent in expressing thanks to the donor countries which have been so kind as to leave these books, which formed part of the book exhibition, for the benefit of the University of the West Indies. I believe that my Professor friend who is on stage would have been the person more competent to express thanks on behalf of the University. But in as much as I happened to represent the Government of Trinidad and Tobago and the Council of the University, may I play this part for my Professor friend, and express on behalf of the University thanks to all the countries that have been so kind in leaving these books for the benefit of the students, who are going to study at the University of the West Indies.

CHAPTER 1

Conference Arrangements

The Background of the Conference

1. The First Commonwealth Education Conference held at Oxford, England, in 1959, was followed, in 1961, by a Conference of Commonwealth experts on The Teaching of English as a Second Language. It was held at what was then Makerere College in Uganda. In view of the success of this Conference, the Second Commonwealth Education Conference, held in New Delhi in January 1962, considered whether, and on what terms, it was desirable to hold conferences of experts in other subjects. The Delhi Conference agreed that this was a suitable form of Commonwealth co-operation in education provided the following principles and criteria were taken into account:

(a) the conference must offer reasonable assurance of providing significant and worthwhile results of value to the participants and to their countries;

(b) it must assist in raising the standards of teaching and education in the specialist field concerned;

(c) in determining priorities, preference should be given to conferences which are designed to meet the needs of developing countries and which concentrate on areas where the need for the training of key people is of special importance; and

(d) care must be taken not to duplicate the work of international agencies and to consider in each case whether a problem requiring attention might be taken up by countries outside as well as inside the Commonwealth.

The Conference further recommended that the Commonwealth Education Liaison Committee should be responsible for considering subjects for further expert conferences, and referred to the Liaison Committee the suggestion that a conference on the teaching of science and mathematics would be of immediate importance to developing countries.

2. Following the deliberations of the Commonwealth Education Liaison Committee, and after consultation with Governments, a conference of experts on School Science Teaching was held at the University of Ceylon, Peradeniya, in 1963. The Report of this Conference was unanimously adopted and commended to Governments for appropriate action by the Third Commonwealth Education Conference in Ottawa in 1964. This Conference added the further criterion that subjects for such conferences of experts should be of clearly defined scope, and went on to recommend that the Commonwealth Education Liaison Committee should consider holding Specialist conferences on mathematics in schools and on the training of technicians.

A Conference on The Education and Training of Technicians was held at the Huddersfield College of Education (Technical), England, in 1966.

3. In 1966, the Commonwealth Education Liaison Committee reviewed all subjects for Specialist Conferences which had been suggested up to that time, either by Commonwealth Education Conferences or by individual governments, and a clear consensus emerged that there should be a conference on The Teaching of Mathematics in Schools, confined to teaching at the primary and secondary levels. It was considered that it would be most useful if some assessment could be made of the different programmes and projects being developed in this field since mathematics forms the basis of science upon which development depends and no country can afford not to strive for better ways of educating its people and for using educational resources to better advantage. It appeared that a Commonwealth initiative in the field of school mathematics would serve these ends. Further, it was the view of professional bodies that new methods in teaching mathematics had reached a stage in their development where Commonwealth educationists would benefit greatly from an exchange of ideas and experience. Accordingly, members of the Commonwealth Liaison Committee, after careful deliberation and consultation with their Governments, agreed that a Commonwealth Specialist Conference of Mathematics in Schools be arranged, that it be confined to teaching at the primary and secondary levels, and that it be primarily a conference of specialists in that field. At the same time, the Liaison Committee set up a Planning Committee to investigate and make recommendations on such matters as the aims, scope and agenda of the Conference, and thereafter deal with its detailed planning, time, location and finance. (Later, when the Commonwealth Education Liaison Unit, which had hitherto acted as the Secretariat for the Commonwealth Education Liaison Committee, became integrated with the Commonwealth Secretariat as its Education Division, financial responsibility passed to the Finance Committee of the Commonwealth Secretariat).

4. Professor W.H. Cockcroft, G.F. Grant Professor of Mathematics at the University of Hull, England, was invited to serve as Chairman of the Planning Committee, and the Director of the Education Division of the Commonwealth Secretariat acted as its Secretary. The original Planning Committee comprised Mr. N.R. Edwards (Australia), Mr. H. Houghton and Mr. D.H. Christie (Britain), Mr. J.C.A. Corea (Ceylon), Mr. P. Cotton (New Zealand), Mrs. L.S. Dorset (Trinidad and Tobago), Mr. A.J. Kisubi (Uganda) and Dr. H.W. Springer (Assistant Secretary-General). Later, as the result of changes in the Australian High Commission and in the British Ministry of Overseas Development, Mr. Edwards was replaced by Mr. P. Bowler, and Mr. Christie by Mr. H.D. Moisley. A panel of mathematics experts was co-opted to advise in their various fields; this comprised Miss E.E. Biggs, Mr. D.G. Chisman, Dr. A.G. Howson, Mr. R.C. Lyness, Mr. J.B. Morgan, Mr. G.C. Nobbs, Mrs. E.M. Williams, and Professor J. Wrigley. During the later stages of planning, and in Professor Cockcroft's absence in the United States, Mr. J.B. Morgan acted as Chairman of the Organising Committee.

5. In considering and consulting with Governments about the venue of the Conference, the Planning Committee and the Commonwealth Education Liaison Committee had regard to the location of previous Commonwealth or Specialist Education Conferences, as geographical and regional rotation in these matters is desirable.

It was also agreed that it would be advantageous if the host country had been engaged in development projects in the field of mathematics, or was conveniently located in relation to other countries engaged in such work. An invitation from the Government of Trinidad and Tobago to hold the Conference in the University of the West Indies, St. Augustine, Trinidad, from 2nd to 14th September 1968, was warmly accepted.

6. In September 1967, invitation to participate in the Conference were extended to all Commonwealth Governments by the Commonwealth Secretary-General, Mr. Arnold Smith, in association with the Government of Trinidad and Tobago. In his letter of invitation, Mr. Smith expressed the aims of the Conference as follows:-

(i) To survey current practice in school mathematical education at primary and secondary level, both in teaching methods and procedures and in content;

(ii) To consider the principles on which contemporary courses should be planned and executed for the different levels of children's development and abilities;

(iii) To examine the problems which must be met if courses are to be appropriate to the differing cultural, scientific, and economic needs of the countries concerned.

7. "Clearly," he said, "the scope of the Conference will need to be limited if it is to achieve its maximum usefulness for the school systems of the different countries of the Commonwealth. It has been decided, therefore, to confine the agenda to the teaching of *school* mathematics. It will restrict its study of the subject to primary and secondary schools, thus eliminating both pre-school and post-secondary education".

Participation

8. The Conference was attended by ninety-four delegates from twenty-nine countries and territories of the Commonwealth. A full list of the delegates and observers is included on Appendix IV of this report.

The Conference Programme

9. The Campus of the University of the West Indies at St. Augustine provided a very attractive setting for the Conference which was opened by the Prime Minister of Trinidad and Tobago, Dr. the Right Honourable Eric Williams, who was introduced by the Chairman of the Conference, Professor W.H. Cockcroft. A vote of thanks was proposed by Dr. H.W. Springer, Assistant Commonwealth Secretary-General.

In conducting the Conference the Chairman was ably assisted by the Co-chairman, Dr. K.A. Julien, Dean of the Faculty of Engineering in the University of the West Indies.

10. The main themes of the Conference and the guest speakers, who presented papers at Plenary Sessions on them were:-

(1) *Fundamental ideas and objectives of mathematical education*
Professor G. Polya (Stanford University)

(2) *The teaching of mathematics at primary level, including both method and content*
Miss E.E. Biggs (Department of Education and Science, London)

(3) *The teaching of mathematics at secondary level – (a) as a general 'core' subject, and (b) as a subject for those with specialist requirements*
Professor W.W. Sawyer (University of Toronto)

(4) *Assessment of children's progress and evaluation of programmes: purpose and method*
Professor J. Wrigley (University of Reading)

(5) *Teachers: Selection; initial and subsequent training*
Mr. D.A. Perera (Ministry of Education and Cultural Affairs, Colombo)

(6) *Resources for learning mathematics, including textbooks, films, radio, television, programmed learning, etc.*
Professor A.L. Blakers (University of Western Australia)

11. Commonwealth Governments were invited to contribute papers on topics related to those dealt with in the lead papers and on their programmes in School Mathematics. A list of the contributed papers appears in Appendix III of this report. In addition, international organisations involved in assisting programmes of mathematics education in developing countries were invited to send observers, and to contribute papers, and two sessions of the Conference were devoted to studying Commonwealth and International Co-operation in this field.

12. After the presentation at Plenary Sessions of the main topics the themes were considered in detail by two Working Groups, A and B, which then reported to the Conference on their discussions and recommendations or suggestions for action. Working Group A concerned itself with primary and secondary education and working Group B with teacher training, assessment and evaluation and resources for teaching mathematics. The programme was so arranged that it was possible for delegates to take part in the work of one Working Group in each category.

13. A notable feature of the Conference was a very fine display of textbooks and teaching materials. The Planning Committee had invited Commonwealth Governments to arrange to send from their countries samples of textbooks and teaching materials used in their schools, and a number of countries responded to this invitation. In addition, Professor A.L. Blakers, of the University of Western Australia, arranged for a wide variety of publishers to send specimen copies of their publications. As a result, one of the most comprehensive collections of school textbooks on mathematics ever displayed was available to the delegates at the Conference. At its conclusion, Professor Blakers, and the representatives of the Governments of Britain and Canada, announced that the books on display were being presented to the University of the West Indies. A bibliography of the books exhibited at the Conference is given in Appendix I.

14. Arrangements were made for delegates to visit a few schools and teacher training colleges, and two afternoons in the programme were devoted to this.

15. The closing session of the Conference was addressed by the Minister of Education and Culture in the Government of Trinidad and Tobago, Senator the Honourable D.P. Pierre.

16. Delegates were impressed by the cordiality, warmth and hospitality they encountered throughout their brief stay in Trinidad. The Pro Vice-Chancellor of the University of the West Indies, Dr. H.D. Huggins, was host to the participants of the Conference on the first day. Later in the week the Governor General of Trinidad and Tobago, Sir Solomon Hochoy and Lady Hochoy received the delegates. A variety show organised by the Ministry of Education and Culture was highly appreciated.

Acknowledgements

17. The Commonwealth Education Liaison Committee and the Commonwealth Secretariat wish to place on record their deep gratitude to all those who contributed to the success of the Conference. While it is difficult to enumerate the persons who have given so much of their time and with unflagging devotion have worked for the Conference, the Commonwealth Secretariat is particularly appreciative of the concern shown by Dr. Eric Williams in the problems the Conference was tackling and the interest and assistance given by the Minister for Education and Culture, Senator the Honourable Donald Pierre. Special indebtedness is acknowledged to the Permanent Secretary for Education, Government of Trinidad and Tobago, Dr. Romain, Mr. Walter Jones, the Liaison Officer of the Conference, the Administrative Officers, Committee Secretaries, and Verbatim Reporters who contributed to the smooth working of the Conference. A special word of thanks goes to the Supervisor of the typing pool and the typists, Transport Officer and drivers for the cheerful willingness and graceful co-operation in meeting the demands of a tight schedule, to the Pro Vice-Chancellor of the University of the West Indies, Dr. Huggins, the staff of the Institute of Education, the School of Engineering and Milner Hall for their personal interest in the welfare and comfort of the delegates.

18. The support given by Commonwealth Governments and international organisations is gratefully acknowledged. The Conference owed much of its success to the skilful chairmanship of Professor Cockcroft, to the co-operation and wise counsel of the Planning Committee, to the discussion stimulated by the high standard of papers delivered by the various experts, and to the exchange of views by delegates leading to constructive thinking and better mutual understanding. Thanks for the preparation of this report are due to the members of the Editorial Committee Professor Cockcroft, Mrs. Williams, Miss Biggs, Mr. Floyd, Professor Wrigley, Dr. Howson and Mr. Lyness, who assisted in compiling the chapters, to Mrs. A. Krishnaswamy, Research Officer of the Commonwealth Secretariat for her work in organising, editing, and proof reading, and in particular to Mrs. E.M. Williams, who as Editor-in-Chief has been mainly responsible for the preparation of the manuscript.

CHAPTER II

The Work of the Conference

Introduction

1. The selection of mathematics as the subject of the 1968 Commonwealth Conference was particularly timely. School mathematics is passing through an epoch of change unprecedented in its range and pace. It involves more than a few innovations. It is a far-reaching movement which affects the role of the teacher, the material to be taught and the nature of the end-product. In some parts of the Commonwealth new schemes of work are already in use, though subject to revision; in some places schemes are being tried out on a limited scale; in other areas the adoption of a revised programme is under discussion. There is still an element of experimentation, and the situation is open. In all countries there appears to be a desire to bring the teaching of mathematics into line with the needs of the day. In such circumstances it seemed to delegates that the Conference could play a useful part in discussing what developments are desirable and the means by which they can be encouraged.

The nature of the changes

2. Before the Conference began, Governments submitted twenty four papers on the present practices and planned developments in mathematical education in their countries. From these accounts it is evident that changes are being considered in all aspects of school mathematics. Of these the most widely influential could ultimately be the decreasing use of direct instruction by the teacher and a greater stress on the creation of classroom situations in which pupils enquire, investigate, experiment and draw conclusions for themselves. Such a mode of learning emphasises both the origins of mathematics in practical experiences and its development through the active thinking of the individual learner. The function of the teacher is more complicated and no less onerous, but delegates endorsed the view that pupils learn more thoroughly and understand more deeply when they explore mathematics in this way. This fundamentally different approach to learning mathematics was initiated by groups of teachers who wished to bring the subject into line with the educational principles underlying the rest of their teaching. Nevertheless, the traditional formality of mathematics lessons is so firmly rooted that it could take a long time to modify. The Conference was so convinced that giving pupils opportunities to use their initiative and inventiveness in mathematical enquiries results in a happier and more confident attitude that the theme recurs throughout this Report. It is taken up fully in Chapter IV and again in Chapter V.

3. The other major change to which the Conference gave detailed attention was the modernisation of the content of mathematics courses. Two distinct pressures were discerned here, one in the direction of relating the syllabus more closely to the needs of industry, agriculture and commerce, the other urging the inclusion of more recent mathematics and a greater stress on structure and logic. This duality will

be seen in many sections of the Report. There need be no contradiction in these two influences; both spring from the wish to suit mathematics courses to contemporary conditions of work and thought. Yet it is not an easy task to give both their due. The problems are faced in Chapter V.

The Scope of the Report

4. In considering the problems created by the tide of change the Conference had to take account of the great diversity in material and social conditions among the countries represented. Early discussions were much concerned with the obstacles which hinder any enlargement of the learning programme in mathematics whether it be an extension of the topics to be taught or the provision of materials and apparatus for pupils' use. Background papers sent in by some Governments made the difficulties dramatically plain. As instances, three very different papers may be quoted.

Professor C.O. Taiwo of the University of Lagos in his paper Primary School Mathematics in an African Society says: " . . . not all Yoruba children attend school. Of those who do, most leave either during or at the end of the primary school course . . . In counting, the Yoruba have a name for every counting number, however large. The name may be long and involved but it is precise . . . The Yoruba have no numerals and no symbols for the mathematical operations so evident in the number words and calculations . . . Strokes are sometimes chalked on the walls to register a count of periodical occurrences . . . In the use of Yoruba as a medium of teaching mathematics . . . there is the problem of evolving a decimal system based on the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9."

Mr. David P. Ambrose of the University of Botswana, Lesotho and Swaziland writes in his paper Mathematics Teaching in Botswana, Lesotho and Swaziland:

"The Republic of Botswana . . . is almost entirely desert or semi-desert with isolated villages near water-holes and boreholes . . . Years of drought have reduced Botswana to extreme poverty. In Lesotho many school-children and some adults have never seen a car or a jeep, though aeroplanes are familiar . . . Lesotho, with the highest literacy rate of any sub-Saharan African country . . . does not have the financial resources for further large scale improvement."

A paper submitted by the Government of India includes these comments: "The syllabus in mathematics at all . . . levels in our country has remained almost completely unchanged since the time when a modern educational system was introduced".

"Teaching aids in middle and secondary schools are practically absent. Even the chalk and chalkboard may not be available . . . If there are any teaching aids in these schools the teachers are not using them . . . In primary schools the condition is a little better, but there also it is equally disappointing looking at their need . . . Many such aids are necessary for introducing the old and new concepts. India is a vast country and the number of primary, middle and secondary schools is about a million."

It should be added that all three papers go on to describe the energetic planning of new mathematics schemes in progress in their countries.

5. In view of the cost of commercially produced equipment Group A.1 gave close attention to the use of local materials in any type of environment but amid such variety the Conference could not work out full details in the time available. Only general guidance could be given on exploiting the resources of the home, the natural surroundings and local usages, thus enabling schools even in the most limiting conditions to embark on a mathematical scheme of active investigation. More sophisticated material requiring organised production is an imperative need and can only be provided in some countries if external aid and co-operation continue to be given. (See Chapters VI and VIII).

6. Defining the scope of the different sections of the Report is difficult in face of the many variations in the organisation of education from country to country or even from one region to another. The minimum age of entry to school may be 5, 6, 7, years or older. The name primary may not be used; if it is, it may denote schooling for any period from 5 to 8 years in length. For its own purposes the Conference defined primary school as consisting of the first 6 years of school. Secondary may follow 'primary' or there may be an intermediate stage sometimes known as middle school. The Conference included in secondary the years at school after primary, that is after the first 6 years. It was expected that Group A.2 would direct its attention chiefly to the period up to a Secondary School Certificate. In the event their discussions were concentrated on the first three secondary years, considering the content of a 'common core' of what is desirable for pupils in general. This limitation of range is significant. There were several reasons for the decision: some areas have an examination at the end of the third year; where some secondary schooling is compulsory, the statutory period is likely to be three years of longer: where it is selective some common ground for various types of school makes for flexibility. It is clear that the intention was to give maximum thought to the years with the largest school population where the problems are more difficult and their solution more urgent.

7. The coming of radical changes in mathematics courses poses the question of their value. Means must be found to assess their results and this is the more urgent when differences between the old and the new programmes are fundamental. Established examinations must themselves be examined and teachers' own testing of their pupils must be reconsidered. When teaching becomes concerned more with understanding and with individual development of ideas, former modes of assessment may be useless. Chapter VI deals with the evaluation of new courses, changes in the form and syllabus of standard examinations and new types of checking on pupils' progress.

8. In all countries the greatest obstacle to progress in mathematical education is the shortage of teachers able to face the new demands. This problem with its different facets in various parts of the Commonwealth was under constant discussion. Inadequate as any suggested remedies may be, the Conference thought it necessary to give all possible help by presenting the many ways in which solutions are being sought, both in countries where all teachers have had professional training and in those areas where unqualified teachers form a large proportion of school staffs. References to this crucial matter will be found in Chapters IV and V. A full treatment is reported in Chapter VII where both initial training and courses for serving teachers are discussed.

9. The greater emphasis on pupils' practical experience as a basis for learning requires that a classroom shall be well stocked with equipment and various sources of information. Under the heading of resources Group B.3 studied the supply of books and materials which should be accepted as essential in all schools. One important factor in spreading new ideas and attitudes is the development of such new means of communication as film, radio, television and video-tape. Chapter VIII examines realistically the possible supply of learning aids where resources are scarce and schools are widely scattered. This theme is developed more fully on a basis of mutual help within the Commonwealth in the chapter on Commonwealth Co-operation in School Mathematics.

Fundamental Ideas and Objectives

10. When changes of such magnitude are taking place on so extensive a scale it seems likely that the ultimate aims of school mathematics are influencing the choice of innovations and in turn are being modified in the process of shaping new programmes.

11. Some of the background papers showed that the objectives of a mathematical education had been carefully considered by the planners of new courses. Two strands can be distinguished amid the variety of statements of aims. Following Professor Polya's analysis in his Lead Paper we can say that the first stresses the usefulness of mathematics; the second looks for some enhancement of the personality of the learner. Dr. Bryan Thwaites, in the paper he submitted on The Way Ahead, hopes "that mathematics will be taught in the future so as to maximise its usefulness in each particular country". It was frequently said during the Conference that each country should select from suggested topics, interpreting them to suit its economic and scientific needs, and should develop a course on the basis of its distinctive culture. Dr. Thwaites goes on to speak of basing "mathematical education on situations which the pupils understand and appreciate from their own personal experience" and giving them "the satisfaction of original and creative work". This theme of individual learning is echoed in the paper from Canada in the statement of the aim of current developments as "adapting curricula, methods and organization to the individual child by means of continuous progress, learning by inquiry and the provision of multiple options". The twin goals of usefulness and individual stimulus are seen again in the objectives set out in a paper from India: to develop problem-solving skills; to help an individual to select a suitable career in life; to develop a healthy citizen useful for home, society and nation; to develop the personality of the child through clear, logical and critical thinking, and the power of concentration. With the developing intellectual powers seen in pupils at the secondary stage, enquiries will often extend beyond immediate practical investigations into problems which arise from mathematical ideas and structures that pupils have already met. The search for solutions to such problems becomes one of the chief means of extending mathematical abilities and of shaping a systematised body of knowledge.

12. Certain principles emerge from these aims and can be seen worked out by the various Groups. Mathematics is for *all* pupils whatever their level of ability or maturity. The limitation of primary courses to arithmetic is fast disappearing and at the secondary stage a full if simplified scheme of mathematics is suggested for the

less able pupils. The method of inquiry and experiment implies the priority of understanding the phenomena and relationships seen in the environment and includes the useful acquisition of insights and skills. It is here that method and content are found to be interdependent. Investigation of the environment involves the method of pupil's discovery and also requires such ideas as sets, relations, graphs and simple statistics. This exploratory aspect of mathematical activity also develops an individual's confidence in his ability to think things out for himself and at the same time shows him the power of mathematics in wider fields. In this way he enlarges his view of himself and of men's achievements in the control and prediction of natural events. Instances of such experiences will be found in the Report of the primary group.

13. The central position given to problem-solving draws attention to the developments that spring from the inquiry method. It implies a search for relations within the situation being investigated; it may mean shaping a hypothesis or hunch; it can lead to devising a programme for finding a solution. Professor Polya draws a valuable distinction between the routine problem, which is essentially repetitive, and the novel problem, which calls on previous experience, orderly thinking and sustained effort.

14. A background paper from Scotland includes among objectives the training of pupils in mathematical language as it is used today. The ability to read mathematics whether in words, symbols or diagrams is given prominence in the Report. It is seen as a requirement for the ordinary citizen; but also the language of mathematics is the means of clarifying a pupil's findings and communicating them to others. In particular the use of diagrams has come to occupy an important place in all stages of school mathematics. As Professor Polya says: "Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits".

The Teaching of Mathematics at Primary Level including both Method and Content

Method

15. It was agreed that the most important new principle at the primary stage is that mathematical education should be based as far as possible on pupils' practical investigation and experiment under the teacher's general guidance. Two reservations were made: basic computational skills must be established and the necessary nomenclature and symbols must be given to the pupils. A movement towards the adoption of more practical methods is under way in some countries. For instance Singapore reports new teaching methods and includes in the primary syllabus 'experience through individual and group activities' with a large range of materials. The West Indies can show developments through such publications as 'Ideas in Mathematics', which "relates many of the 'New Mathematics' concepts to concrete situations" in Trinidad and Tobago; Barbados has produced a Curriculum Bulletin, 'Junior School Mathematics' which presents articles on Modern Trends, Hints and Suggestions for Teachers. Malawi is experimenting with a scheme which draws on any material relevant to Malawi conditions and traditions. India has a project which aims "to give training in recognition of patterns in number and space, to popularise the use of Cuisenaire rods, geo-boards, flannel graphs, and other teaching aids". Australia and

Canada both report pilot projects. But the papers from a majority of Governments showed that progress is slow. This is due in part to a strong tradition, sometimes linked with a selection examination, and to shortages of space and materials; but it is mainly because many teachers are not familiar with this type of classroom programme and are not yet ready to effect the change. The reports from some countries suggest that new plans to revise the primary mathematics curriculum are concerned chiefly with introducing some of the more abstract of the new ways of teaching number and take little account of the pupils' need for practical activities. It was therefore decided to discuss fully and to report in some detail on the necessity of personal experience as the foundation for children's thinking and on the opportunities for such experience to be found even in difficult conditions.

16. The Report contains suggestions on making the transition to this kind of programme which should help enterprising teachers; but delegates made it clear that much has to be done by way of teachers' courses, visits to schools, discussions among groups of teachers, etc. if this important change is to gain momentum. In a useful list of classroom requirements local and improvised materials are a prominent feature. Yet these alone cannot be sufficient; such things as measuring instruments, various types of graph paper, books for reference, etc. are also necessary in an investigational programme.

Content

17. Active learning methods can be used whether the mathematics to be learned is new or traditional. The traditional content has a long history and new fundamental ideas are likely to spread only gradually. As a paper from Britian indicates, it is only recently that programmes with a new basis have been fully worked out in that country for the primary years. The early stages of the new foundational work in number are discussed fully in Chapter IV.

18. Through the handling of everyday objects and any available structured material, a sequence of development from the sorting and matching of pre-counting days leads by way of sets and correspondences, relations and sequences to counting, notation and operations. As was brought out vividly in Miss E.E. Biggs' Lead Paper, the search for pattern is the starting point for the growth of the idea of a function. Tabulations and graphs are suggested as aids to pupils' recognition of relationships. The use of the idea of movement on the number line was considered and was related to transformations, which are also approached through pattern-making. The value of various kinds of diagrammatic representation of mathematical relations from the earliest days was stressed by delegates. This was linked with the study of movement, change and growth.

19. The lightening of the computational load when metric measures are used was welcomed, particularly because it allows more time for other studies. For instance, the growth of geometrical ideas through constructional activities was worked out by the group. Such topics as statistics and graphs arose in considering environmental studies and the place of mathematics in relation to other subjects of the curriculum.

20. Delegates were impressed with the need for a broad programme of this kind but they realised that many primary teachers know little of some of the mathematical

ideas and techniques which have been put forward and found to be within the scope of primary pupils. Courses to deal with this new material are required both in Teachers' Colleges and for teachers in service.

21. **Suggestions for Action**

(1) *All those concerned with education should be made aware of the new developments: supervisors, head-teachers, lecturers in Teachers' Colleges, and parents. They can be given opportunities to try out the new ideas for themselves.*

(2) *A programme to introduce teachers to modern ways of learning and to give them experience of learning new mathematical topics by active means is essential.*

(3) *The new kind of programme demands more and different basic materials and equipment, some of which will need large-scale local or commercial production.*

(4) *A local teachers' centre, set up in a school or college, can provide opportunities for teachers to discuss classroom experiments, to make some of the things needed for children's activities, and to share ideas about the use of local materials.*

(5) *In planning new school buildings it should be borne in mind that a programme of children's practical activities requires different kinds of furniture, equipment and buildings: for example, flat-topped desks for all grades, extra tables, shelves and storage cupboards (steel where necessary), display space for children's work.*

(6) *Where the mother tongue used for teaching in the early years does not contain words required in mathematical experiences the necessary vocabulary should be given in the ultimate language of instruction.*

(7) *Where it is still necessary to select for secondary education by an examination, the form and scope of the examination should take account of the new emphasis on understanding, on individual learning and on non-arithmetical content.*

(8) *It is hoped that countries which do not use the metric system will make definite plans for its adoption.*

The Teaching of Mathematics at Secondary Level

22. It is not only in the length of secondary courses, in the proportion of children for whom secondary education is provided and in the examinations taken that there is wide divergence from one country to another. The extent to which mathematics courses have been modernised varies too. Background papers show that a few countries are still at the planning stage; some have well-established projects; the majority have new courses in progress at a limited number of schools. In general the School Mathematics new schemes derive from projects initiated in Britian or the U.S.A. or from the Report: New Thinking in Organisation for European Economic Co-operation. In several territories, for instance in Australia, Canada and Ghana,

individual schemes have been produced. In the search for common ground amid such variety, Group A.2 decided to discuss particular topics in the expectation that general principles and suggestions for the future would emerge.

The transition from primary schools

23. Consideration was given first to the problems created by new primary mathematics programmes. As Professor W.W. Sawyer says in his Lead Paper, "secondary teachers should be preparing for a steady transfer of algebra and geometry to primary schools". The maintenance of continuity in methods of approach and in subject matter was delegates' main concern. Any entrance examination must take account of recent developments.

Planning a new syllabus

24. When syllabuses are being reconstituted, freedom of choice for the teacher is important; but it was recognised that many teachers are not yet ready to take this responsibility. Widely based syllabus committees with substantial teacher representation were thought to be desirable.

25. The wide range of ability at the secondary level demands a range of teaching methods. It was suggested that each year of pupils could be arranged in 'sets' according to ability, and taught at the same time. This allows more practical methods to be used for the less able groups and makes any necessary transition easier. Suitable provision for practical work is clearly required.

Essential elements of general mathematical education

26. The possibility of a common core in mathematics for the first three secondary years was carefully examined. It was stressed that only a skeleton of mathematical ideas and procedures could be given since it is essential that each country should plan courses suited to its own needs and resources. On the foundations of a common core, courses can be devised to match the abilities of pupils, both the academic and the more practically minded. On the one hand the courses must be related to the various types of job that are now available or are likely to be created by industrial and agricultural advances in the next few years. On the other hand the basic content in the early years can be sufficiently alike to enable transfer from one course to another to be made smoothly. There was a strong plea from some delegates that the underlying mathematics should be the same in all Commonwealth countries. Group A.2 submitted a possible common core in which traditional and modern elements are fused.

Courses to School Certificate level for those with specialist needs

27. For the fourth and fifth secondary years suggestions were made for the inclusion of modern topics which have social and economic values as well as those needed for the biological and physical sciences and for future technicians. It was emphasised that a wide programme is necessary because for many pupils mathematical education will cease at this point.

Higher level secondary work

28. In the stage beyond School Certificate the demands of specialists in many fields need to be met, including those for whom mathematics will be a main study.

Stress was laid on the importance, at this level in particular, of the development of logical and systematic thinking and the tactics of problem-solving. On this basis the principles on which the subject-matter should be chosen and treated were set out rather than details of possible courses.

Flexibility

29. The employability of secondary school leavers was one of the main themes in discussion. It was stressed that flexibility in planning courses was all-important if employment demands are to be met and all pupils found jobs. Equally important in an epoch of change is the need to produce flexibility of mind in secondary pupils through their study of mathematics.

30. **Suggestions for Action**

(1) *Links between primary and secondary schools should be established to ensure that transition is as smooth as possible.*

(2) *Teachers should be given considerable freedom in deciding syllabus content. It is often desirable to set up a syllabus committee on which teachers play a major role and both industry and Government are represented.*

(3) *Secondary courses should be relevant to the various mathematical needs of the community in which they operate.*

(4) *In the first three years of secondary schooling mathematics courses should have a common core; thereafter they should be varied to suit the needs of particular groups.*

(5) *Mathematics and other subjects of the curriculum should be inter-related so that the contribution of each subject can be used to the best advantage.*

(6) *Timetables should be planned to allow simultaneous teaching of different ability groups within a particular year.*

(7) *To facilitate group working and practical classroom activities, suitable furniture should be chosen.*

(8) *It is essential to provide a mathematics room and, where possible, a suitably equipped mathematics laboratory.*

Assessment and Evaluation

31. One of the effects of the current changes has been to stimulate assessment of new programmes in regard both to the individual pupil's progress and to the success of a programme in achieving its declared objects. The group which discussed the problems of assessment agreed on two principles. In any scheme of testing the welfare of the pupil must be the first consideration. The effects of the testing situation must be *beneficent,* causing no hindrance to the pupil's mathematical or personal development. Secondly, the teacher's role should be enlarged to include the detailed day by day recording of each pupil's work, necessary in more individualised programmes, and also substantial contributions to various external forms of assessment.

32. Many forms of assessment are now available. These are described and useful examples given in Professor Wrigley's Lead Paper. The selection of a test for any occasion depends on the purpose of the assessment. Tests may be required to assess the level of a pupil's understanding; a situation test of the Piaget type may then be chosen. Examinations to determine the achievement of a candidate will require different test items from a selective examination. The group discussed the particular uses of multiple choice, multi-facet, open-book and other types of item.

33. Evaluating new modes of learning and new content is the subject of many investigations. For any project delegates agreed that aims must be stated before any assessment could be planned. Comparison of the results of different projects is being undertaken in several countries where project schools and control schools can be tested and compared.

34. Delegates emphasised that very many teachers have little knowledge of modern testing techniques and urged the need for further training if they are to be ready for their new responsibilities in testing and in providing assessments for external uses.

35. **Suggestions for Action**

(1) *a. The understanding of mathematical language should be tested, among young children in particular.*

b. The language used in verbal tests should be such as the pupils can be expected to understand.

(2) *Constant evaluation of each pupil's progress and the suitability of the programme should be made by a teacher and appropriate records kept.*

(3) *Teachers need to use a variety of forms of test in assessing their pupils and should be given help in acquiring the techniques.*

(4) *The objectives of a programme should be clearly defined so that they may form the basis of assessment. Comparisons of results from similar programmes should be encouraged.*

(5) *Where a selective or an achievement test is necessary, it should include multiple-choice, multi-facet and traditional types of item, and where possible an assessment of course work and a practical test. A high rating should be accorded to the opinion of a teacher who has the necessary skills in assessing.*

(6) *When an externally set test has to be used teachers should play a substantial part in devising the content and method of the test.*

(7) *Further opportunities should be found for sharing experience in assessment and evaluation between countries of the Commonwealth.*

TEACHERS, Selection; Initial and Subsequent Training

Supply

36. From all parts of the Commonwealth came evidence of a serious shortage of competent teachers of mathematics. In some instances the deficiency was in qualified

teachers, particularly in primary schools. Some countries had a large number of untrained primary teachers because money was not available to pay the salaries that would be due to such teachers if they were trained. The Conference strongly deprecated the general employment of unqualified teachers. Everywhere too few teachers have sufficient knowledge of the topics and methods now being introduced as essential elements in a modern mathematics course. Group B.2 recognized the drain of mathematically trained personnel away from teaching to careers in industry, administration, etc. and delegates were particularly concerned that teaching should be accorded the status and rewards that would enable the profession to attract a higher proportion of able mathematicians into its ranks.

Selection

37. New modes of learning, in primary schools particularly, require some personal qualities that were not thought to be requirements in more formal teaching. But, as Professor Perera pointed out in his valuable Lead Paper, we are still remarkably ignorant of the qualities that make a good teacher; we cannot therefore identify them accurately in candidates for training. An attempt to form a judgement is usually made at an individual or group interview and experienced selectors place reliance on such interviews as providing necessary evidence to place alongside examination qualifications and school records. Candidates for primary training may have scanty knowledge of mathematics but this can usually be remedied during training. For secondary teachers delegates regarded a sound knowledge of mathematics as essential; yet they were agreed that candidates with only moderate mathematical qualifications could make valuable teachers.

Initial training

38. Since all primary teachers have to teach mathematics Teachers' Colleges should provide for all students a substantial course in mathematics which would give them a fundamental understanding both of the mathematics they will be expected to teach and of the value of learning through investigation and experiment. Students with a special interest in mathematics should be offered a further course so that they develop some expertise in this field and in due time may become leaders. Qualifications for secondary teachers fall into two main types: a degree followed by professional training, and a degree incorporating training. The extension of opportunities for degrees with a bias towards education was welcomed by delegates; they regarded professional training as essential. The supply position could be improved if some Teachers' Colleges could offer secondary courses to suitable candidates without a university degree but with promising mathematical ability. Clearly the colleges need to see that tutors have adequate knowledge of new developments and that their facilities in buildings and equipment are suited to modern needs.

Subsequent training

39. The urgent need to provide mathematics courses for serving teachers is well expressed in a report from Quebec (Canada). "Of all the subjects in the (curriculum), mathematics probably requires more than any other a profound redevelopment of the teaching aspect. The change needed is so thorough that, to those who have studied traditional mathematics, it will appear as though a new subject has been

included". To deal with the situation this Province describes an intensive programme of further training, expanding over a period of several years. The problem is being tackled vigorously and in many ingenious ways in those countries which have the personnel to embark on large-scale programmes. The courses vary from the widespread one-day or two-day sessions reported in a paper from Victoria (Australia) to three- or four-week courses for smaller numbers of teachers. Delegates were impressed with the part-time courses that could be held at universities and teachers' colleges and fitted into teacher's timetable. Workshops, correspondence courses, television, radio and local mathematics centres were among the means being tried out. The number of teachers involved is immense and inter-Commonwealth aid will be needed on a correspondingly larger scale.

40. **Suggestions for Action**

(1) *Efforts should be made in each country to examine further the reasons for the shortage of competent teachers of mathematics and practical steps should be taken to ensure and retain a good supply of able teachers in the profession.*

(2) *Entry qualifications to Teacher Training Colleges should ultimately be the Secondary School Certificate or several passes in the General Certificate of Education or its equivalent.*

(3) *The final selection of entrants should be in the hands of the Training College or Education Department.*

(4) *For all students in Training Colleges there should be a mathematics course which should give them insight into both new content and new methods of teaching. Some colleges should provide courses for students intending to teach mathematics in secondary schools.*

(5) *Opportunities should be given to college tutors to keep abreast of modern developments through attendance at courses and study leave at home and overseas.*

(6) *In planning new buildings for colleges and other training institutions provision should be made for mathematical laboratories and equipment.*

(7) *Teachers with little knowledge of mathematics should be helped at local centres and special courses. Universities, colleges and professional bodies should be asked to assist in the provision of courses.*

(8) *Universities and Training Colleges should be encouraged to provide evening courses in mathematics.*

(9) *Facilities for further training in mathematics and in modern teaching methods should be made widely known.*

Resources for Learning Mathematics

41. In an age of technological developments in means of communication it is worth noting that the Conference was convinced that the teacher is the most important resource and "the main channel for the communication of mathematical ideas to the student", as Professor Blakers says in his comprehensive Lead Paper. He

also speaks of teachers who are excited about their subject and communicate this excitement to their pupils. The Report of Group B.3 shows that the pupil's response was their main criterion as different resources were considered.

42. Professor Blakers and the delegates also agreed on the high rating to be given to books as a resource for teachers as well as pupils in the learning situation. The value of a good accessible collection of books was stressed repeatedly and high praise was given to the outstanding book display that was arranged for the Conference. The Bibliography on p.159 gives a list of the books displayed and includes Prof. Blakers' recommendations of books that should be readily available for teachers to consult.

43. The uses of photographs, slides and overhead projectors were discussed; critical consideration was given to recent experiments with television, films and radio. The still picture and the radio voice were thought to be generally too limited to be mathematically interesting. Such important elements in mathematics as movement, change, sequence and development are not easily conveyed by these means. More success can be obtained with television, films and an overhead projector where sequences can be built up. Such learning aids are costly though they might be valuable in countries where good teachers of mathematics are in very short supply. The extension of their use must depend on financial resources. A television programme achieves interesting results when teachers' classroom contributions, pamphlets and notes are used to supplement the broadcast.

44. Programmed learning as a mode of individualised study was seen as having many forms, both old and new. The recent attention given to it extends beyond programmed textbooks, correspondence courses and work-cards to machines which are at present too costly for general use. The value of such tightly programmed devices was thought to lie in remedying individual gaps in knowledge and in developing specific techniques and skills. Designing programmes is the greatest difficulty which has still to be overcome.

45. Low-cost aids have a universal importance. Chapters IV and V contain many suggestions. Group B.3 gave its attention to essential forms of equipment and the priorities which should govern the production and selection of aids at primary and secondary levels.

46. **Suggestions for Action**

(1) *Modern methods of teaching, using all available means, should be developed alongside the modernisation of content.*

(2) *Textbooks should be provided (and perhaps written) to suit particular areas. Collections of books of many kinds should be provided and countries should co-operate in arranging interchanges.*

A good collection of books should be circulated among Commonwealth countries.

(3) *Authorities should encourage local production of aids and equipment such as types of graph paper, geoboards, wooden models.*

(4) *Each country should develop a multi media approach to visual aids, making the fullest use of such aids as become available.*

Commonwealth Co-operation in Education

47. The last topic to be discussed by the Conference was Commonwealth Co-operation, fittingly, since this theme had recurred in every session when future developments were being considered. Chapter IX sets out in detail the impressive record of programmes of co-operation in Commonwealth countries. The experience gained in such co-operation was an important element in the planning of the Conference. The intention was to bring together representatives from all parts of the Commonwealth to exchange information and opinions on the present position of new mathematics curricula. From such interchange and the discussion of problems that have arisen in attempting the reform of school mathematics, it was hoped to add strength to the movement for reform and to discover how mutual help could best be given.

48. Two major results seem to have been achieved. First, delegates came to know the conditions, the difficulties and the possibilities in mathematical education in countries very different from their own. From this new knowledge can come a fuller participation in aid based on deeper understanding. Secondly, delegates came to the Conference with different views about the changes in mathematics programmes. Some people regarded the study of mathematical structures as the essential element in new schemes; others thought that the main objective was to secure a basis of individual practical experience from which mathematical ideas would emerge. Conference discussions showed that these two points of view were not incompatible. They were seen to be complementary, and it is on this enlarged concept of what the 'new' mathematics means that future planning should be based. The practical and intellectual values of the subject can both be realised to the economic advantage of the countries concerned and the greater appreciation of mathematical thinking among their peoples.

49. Further developments in co-operation, both in carrying through improvements in mathematics teaching throughout the Commonwealth and in providing aid for countries with inadequate resources, were considered to depend on some central agency which could readily put countries into touch with one another in connection with any topic or enterprise of common concern. Delegates expressed strongly the hope that the Commonwealth Secretariat would be enabled to undertake this important function.

50. **Suggestions for Action**

(1) *To encourage the spread of the most effective current trends in mathematical education, definite plans should be made for inter-Commonwealth exchange of information, publications and personnel.*

(2) *The co-operation of all Commonwealth countries in promoting inter-communication should be sought and the Commonwealth Secretariat be enabled to provide the necessary facilities.*

(3) *Professional Associations of mathematics teachers in the various countries should establish closer links, perhaps through a system of reciprocal affiliations, and thus become agencies through which the exchange of new ideas and the results of experiments could take place.*

(4) *In addition to the bilateral schemes now operating, countries with the required resources should co-operate with one another in providing aid in material and personnel to a developing country which is planning to inaugurate, or carry further, a new programme in mathematics entailing extensive courses for teachers, and additional accommodation, material and books for the schools.*

(5) *The role of the Commonwealth Secretariat in implementing the suggestions made by the Conference will be an important factor in the improvement of mathematical education in the Commonwealth.*

Conclusions

There was general agreement that the chief values of the Conference had been

(1) the delegates' increased appreciation of the achievements and problems of countries other than their own

(2) the realisation of the unanimity of delegates about the elements of mathematics it is desirable to include in mathematics programmes, and about the advisability of pupils' learning them in relation to their own surroundings and the needs of their country

(3) the awareness of the benefits to be gained from hearing at first hand about the useful diversities to be found among the countries represented at the Conference.

CHAPTER III

Fundamental Ideas and Objectives of Mathematical Education

Lead paper by Professor G. Polya, Ph.D. (Maths),
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I. Introduction

1. I am deeply indebted to the organizers of this Commonwealth Conference on Mathematics in Schools for having invited me to present the lead paper in this first session. I regard this invitation as a great honour, a great challenge, and a great opportunity. Yet, I must confess, I am also very much embarrassed by it. I see great difficulties inherent in my task and I don't know whether I am good enough to master them.

2. First of all, the aim of this conference is eminently practical. Each of us has urgent practical problems at home, and expects some contribution from this conference to their solution. Yet the theme of today's session is: "Fundamental ideas and objectives of mathematical education". You may have the impression that this theme is too remote from the concrete practical difficulties you have at home.

3. Secondly, I am embarrassed because you may expect me to tell you about the latest results of the science of education. Now, I must confess that I do not believe that there is such a thing as "the science of education". In my opinion teaching is not a science but an art. Or, let me put it a little more carefully (there is no time for a very careful statement): Teaching is, for the time being, much more an art than a science. Yet, if this is so, I cannot tell you scientific truths, just my personal opinions and it is embarrassing for a mathematician to assert things he cannot prove.

4. Here I am, however, and, as far as I can see, I can do nothing better than tell you my personal opinions which I acquired in doing mathematics, in teaching mathematics, and in thinking about the ways of doing and teaching mathematics for a good bit more than half a century. I would be glad, of course, if I could find among you kindred souls who hold similar opinions. Yet it is, in fact, more important to rouse those among you who have different opinions, because the task of this opening address is to start a debate.

5. I must discuss generalities, but I shall try to avoid empty generalities and keep as close to more concrete practical questions as I can. Generalities are needed to put the details into the right perspective. I wish to present you a "philosophy", but I shall emphasize simple points and obvious common sense. After all, I am not quite a philosopher. Do you know what a philosopher is? A philosopher knows EVERYTHING but nothing else.

II. General objectives of the School

6. The family sends the child to the state supported school. What should the school do? The state, the family, the neighbours, the public opinion, everybody agrees, sometimes even the child himself: The school should enable the child to have a job, to earn a living. So it is in the United States, in other highly industrialized capitalistic countries, even in communistic countries.

7. In any social system the child should develop into an adult who can take care of himself and is well adapted to the community. The task of the school is to *contribute its share to the child's development.* In a very primitive and very stable community there is no need of schools: The child is sufficiently educated at home and by unplanned contacts with other members of the tribe. In general, the more complex the community's economic structure and the more rapid its technological or social change, the greater becomes the share of the school in the child's education. In the United States there is a definite trend: The young people stay in school longer and longer and the taxpayers vote higher and higher sums for the schools. And we can observe a similar trend almost everywhere.

8. To turn the child into an employable adult is a crude and narrow aim. In fact, if we conceive this aim too narrowly we are almost bound to miss it. Observe that in a complex economic system there are many different kinds of jobs. The individual should find the job for which he is the most suitable; this is not only in his interest but also in the interest of the community. Yet, to choose the most suitable he must know all the possibilities and so he must have some knowledge of the whole world around him, some sort of general culture.

9. Moreover, to fill his job well, he must be well developed. Yet we do not know in advance what kind of job the growing child will eventually have, and so we must develop him as far as possible in all respects.

10. And so, beyond the crude and narrow objective "turn out employable adults", there appear higher and wider objectives of the school: *Develop all the inner resources of the child. Give him general culture.*

11. I think that the old Greek philosopher Plato would not disapprove of these wider aims. I hope that you don't disapprove of them.

III. Narrow objectives of mathematical education

12. In an economic system that is above a certain primitive level analphabets are scarcely employable. Hence there arises the obvious need of a primary school that should teach every child the three R's, reading, writing, arithmetic. Let me use (in this general introductory talk) the term "primary school" in a not too sharply circumscribed sense which is roughly the following: The school that every child of the age 6 to 12 should attend. To teach the rudiments of arithmetic (the natural numbers and the four basic operations with natural numbers) is an obvious crude minimum task of the primary school. Let me add two items which are almost as necessary and then we arrive at a somewhat enlarged, but still very modest aim of primary mathematical education:

Primary narrow objective:

(1) Arithmetic of natural numbers (+, −, ×, ÷)
(2) Length, area, volume
(3) Fractions, percentage

Also the term "secondary school" should be used in this talk in a not too sharply defined sense: A school that some children of the age 12 to 18 should attend. And let us consider a corresponding modest aim of mathematical education:

Secondary narrow objective:

Professional preparation of prospective technical personnel (technicians, engineers, scientists, managers).

13. Let me emphasize that the two objectives just formulated are severely restricted, narrow, minimum goals, dictated by obvious economic needs. I think that there is little doubt that the schools should attain these narrow objectives. Yet I think too, and I hope that most of you will agree, that the schools should do more and also attain some wider and higher objectives.

14. Let me point out that those narrow objectives themselves may lead to certain wider objectives.

15. There is no doubt that the primary school should teach arithmetic. Now, for all practical purposes, it is sufficient that people do their arithmetic *mechanically;* only some speed and accuracy matters. Yet the school should do more: We should teach the children to do their arithmetic *insightfully*. Why? Because teaching them so we may get results faster and more permanent results. Insightful knowledge is a more ambitious aim than mechanical knowledge. Yet, in teaching as in other activities, the more ambitious aim may have more chances of success.

16. Now, let us look at the secondary school. Mathematics above the primary level is strictly mandatory, *immediately* necessary only for prospective users of mathematics, for certain kinds of technical personnel. Yet when the child enters the secondary school we don't know yet whether he will or will not exercise a technical profession later and so we are obliged to teach mathematics to all entrants. Now, it would be a sad thing if the future non-technicians, who may be the majority, would derive no benefit from years of mathematical study. Therefore we must find a wider objective for the teaching of mathematics in the secondary school so that it should offer something to both kinds of students, to those who will, and to those who will not, use mathematics in their later studies or profession.

IV. General Wider Objectives

17. I have tried hard to show you that starting from narrowly restricted goals of education we may arrive at wider objectives and higher ideals:

Serve the individual and the community.

Develop all the inner resources of the child.

Several such general objectives of education have been proposed, from Plato downward, in various formulations. Here are a few:

General culture.
Discipline of the mind.
Desirable habits of thinking.
Mental and emotional maturity.
A well balanced personality.
Which one of these aims do you prefer? Which one deserves to be preferred?

18. I don't know. I think that all these aims are respectable, I believe that they are all desirable, but I don't know which one is the most desirable – and if I did know I would not take the time to discuss it here. Because, up to a certain point, it does not matter. These objectives certainly overlap a good deal and even those sounding rather different may lead, if reasonably interpreted, to the same practical consequences.

19. Each aim mentioned embodies some high ideal worth striving for. Yet every one may be worthless if you pay only lip service to it, and it may be worse than worthless if it is used as a cheap and empty slogan by pretentious incompetents who are usually the more pretentious the more incompetent they are. Each of those ideals is worthwhile *provided* that you honestly believe in it. And it is really valuable if you earnestly try to live up to it, if you translate it into the everyday practice of the school.

20. Your ideal, whatever it is, should be somehow present in your mind whenever you are planning the curriculum, or preparing yourself for the next lesson, or choosing a problem for homework. Does the item you are considering contribute to some practical, clearly defined narrow objective? Or does it contribute to some wider objective such as general culture or mental discipline? If it does not, why should you bother or why should you bother the children with it? A child in your class or his father could ask you: "But, teacher, what is it good for?" Can you answer this question to your own satisfaction?

21. My opinion is: Have some ideal, see the details of your profession in the light of your ideal, and realize that any detail may raise the question "What is it good for?" Then you will become a better teacher.

V. **On learning: Two simple rules**

22. I have to say now a few words on the process of learning. This is a subject about which old philosophers and modern psychologists said and wrote and printed more than any one of us could read. Yet whatever we read we should read with attention and accept only what we can confirm on the basis of our own well digested experience.

23. I wish to concentrate upon two points which a teacher should never disregard and which may easily arise in our later discussion. They are simple, can be formulated in everyday language, they are essentially classical, and pretty generally, if not universally, accepted. I will call them "principles of learning" but if you wish to call them "rules of thumb" instead of "principles" I shall not be offended at all.

24. (1) *Active learning.* It has been said again and again that learning should be active, not merely passive or receptive. It has also been said, along the same line, that the best way to learn anything is to discover it by yourself. This idea is very old, it can be traced back to Socrates who expressed it by a picturesque metaphor: The ideas should be born in the student's mind and the teacher should act only as midwife.

Yet time is limited, the teacher cannot wait when the labor of childbirth is too long and difficult. Moreover, we cannot expect that high school kids will rediscover the whole of human science, and so we must settle for something less: *Let the students discover by themselves as much as feasible under the given circumstances.*

(2) *Consecutive phases.* It has been said again and again that, in learning, things should come before words, the concrete before the abstract, doing and seeing before verbal expression, and so on. I think that you can find quotations to some such effect in the writings of all famous educators, from Comenius to Montessori and after, but let me quote just one, a sentence by Kant: *Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and ends with ideas.*

It seems to me that this sentence suggests at least some, if not all, of the more important priorities to which we, as teachers, must pay attention. Yet let me paraphrase it, restate it in more down-to-earth language: *Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits.*

I hope that this sentence touches the right chord with you. To explain it thoroughly would need close consideration of several examples for which we obviously have no time. I believe, however, that the two rules formulated here can yield valuable guidelines to teachers who understand them seriously, intimately, on the basis of their own well considered personal experience.

VI. On problems, problem-solving, and the tactics of problem-solving

25. I expect that you have essentially known all or most of the objectives of education and mathematical education and also the principles of learning I have mentioned so far, but I thought that it might be useful to remind you of them and emphasize such points as may arise in one way or another in our later discussions.

26. I am glad that I can say something less well-known and more personal about the next topic.

(a) From any one of the general objectives mentioned you can get down to concrete details of the curriculum in many different ways: There is no scientifically guaranteed best way of teaching, there are as many good ways of teaching as there are good teachers.

Yet there is one specific point about which there is little dissension: All experts agree that mere possession of information is of little value in mathematics. To know mathematics means to be able to *do mathematics:*

To use mathematical language, to find the unknown, to check a proof, and so on. Therefore, to teach mathematics we must give opportunity to the student to do mathematics.

There are many ways of doing mathematics but, from several viewpoints, solving problems appears as the most cardinal mathematical activity. "The solution of problems is the most characteristic and peculiar sort of voluntary thinking", wrote the renowned American psychologist William James. In fact, solving problems may be considered as the specific achievement of intelligence, and intelligence is the specific gift of mankind. Adults are working at their problems and worrying about them, and children's play, which is an anticipation of adult life, is often a sort of problem-solving. Solving mathematical problems at their level comes early and quite spontaneously to some children, and I think that all children could be made ready for it pretty early by the right approach.

At any rate, higher mathematical activities (such as framing new concepts, building theories, constructing axiomatic systems) presuppose considerable mathematical experience which must be acquired mainly by solving problems.

Hence it is well justified that all mathematical textbooks, beginning with the Rhind papyrus, contain problems. The problems may even be regarded as the most essential contents of a textbook and problem-solving by the students as the most essential part of mathematical instruction.

(b) Yet "problem-solving", which became a sort of slogan recently, means different things to different people.

One thing is to teach a specific procedure to solve a specific kind of problem, for instance the usual formula to solve quadratic equations. This may be useful for certain students who need such specific skill in their future profession or in later studies, perhaps for an examination – yet it is hardly useful *in itself* for the general student. Why should the future lawyer or the future truckdriver solve quadratic equations?

A very different thing is to let the students do mathematical problems in the hope that desirable mental habits will result from their work, such as orderliness, precision, ability to concentrate, ability to handle abstract concepts, to name some of the more popularly known objectives of this kind.

There are problems and problems. We must be especially concerned here with the difference between routine problems and non-routine problems.

A routine problem has a specific aim, it should teach the student to use correctly this or that particular rule or procedure or definition, it offers drill and practice, and it does not demand any invention or originality. To solve a quadratic equation with given numerical coefficients is a routine problem for a student who has been shown the general formula.

On the contrary, a non-routine problem challenges the student's inventiveness and originality, it should aim at some more general and higher objective, and I believe that no higher objective of secondary mathematical education can be attained without the judicious use of non-routine problems.

I must confess to you that I feel uncomfortable when I have to listen to a speaker on problem-solving who does not discriminate between routine and non-routine problems. And I feel particularly uncomfortable when the whole behaviour of the speaker arouses the suspicion that he has never solved a non-routine problem himself.

Yet I may be prejudiced: Non-routine problems are crucial for that objective of secondary mathematical education that I personally prefer to all others. My favourite aim is: Mathematical problems should be used to implant in the students' minds whatever attitudes and procedures may be generally useful for solving any kind of problem, the *tactics of problem-solving.*

(c) I have now arrived at my favourite topic. Of course, I would like to talk about it a lot, but there is time only for a brief and rather imperfect sketch.

In teaching problem-solving in the mathematics class we quite naturally come across attitudes and patterns of thought, the usefulness of which is not restricted to mathematical problems.

You all know that widespread bad habit of students: When they have to solve a problem, especially in an examination, they dive into computations or constructions without having quite realized what the proposed problem actually is. Yet the right attitude is just the opposite: First of all, *understand* your problem. Distinguish its principal parts and see each of them as clearly as you can. Try to foresee the result – there is perhaps an easy way to estimate the unknown. Yet, above all, try to *conceive a plan* before you dive into details.

What I have tried to describe here is an attitude the usefulness of which extends far beyond elementary mathematical problems, yet we can impress it on the students' minds in the mathematics class. This attitude (which would deserve a more careful and more detailed description illustrated by example) belongs to the tactics of problem-solving.

I have trouble tearing myself away from this topic. Yet let me just mention a problem-solving procedure, developed on elementary mathematical problems, the interest of which, however, extends far beyond them. Its discovery was attributed to the philosopher Plato by some ancient authors. It was certainly developed by the Greek geometers who called it "analysis" which means "solution backward" in Greek. As the word "analysis" is used today in several different meanings it is preferable to use some other term such as "regressive argument" or simply "working backward". The procedure is mentioned in most older textbooks of geometry: We begin by "taking for granted" the result, the conclusion that we have to prove or the figure that we have to construct. Then we derive from it some other conclusion or figure, hence still another one, and so on, until we arrive at the hypothesis or the data proposed. A fuller description illustrated by appropriate examples would show that this procedure of "working backward" is by no means restricted to geometry, but has a wider interest; it belongs in fact to the tactics of problem-solving.

The foregoing examples, to which I am not allowed to add more here, should give you a first idea of the "tactics of problem-solving".[1]

It is my conviction which I wish to express here as strongly as I can: The teaching of mathematics in *the secondary school should emphasize the tactics of problem-solving.*

Some of the students will, others will not, use mathematics after leaving the secondary school. For users of mathematics the tactics of problem-solving may be the beginning of their professional attitude. For non-users it may easily be the most useful thing that remains with them from the mathematics of the secondary school.

There is much more to say about problems, problem-solving, the tactics of problem-solving, and the relation of these things to the general objectives of mathematical education and the curriculum. Perhaps there will be an opportunity later, in a working group, for a conversation on the details of certain questions raised.

VII. Conclusion

27. The time has arrived for looking back at the discussion of the past hour.

I have talked to you about problems, problem-solving, and the tactics of problem-solving. Yet I did not urge you to adopt my views. If there was anything in what I have said that will start you reconsidering this or that point of your teaching it will be enough and a great satisfaction to me.

28. My views on problem-solving in the schools spring from a general outlook which you may call my "educational philosophy". It is a very informal, down-to-earth philosophy and I presented it, or, rather, hinted it, to you very informally just "between the lines". And certainly I did not urge you to adopt it. In teaching, you see, as in many other things, it does not matter much what your philosophy *is*; it matters more that you *have* a philosophy; but it matters very much whether you do or don't live up to your philosophy.

29. May I repeat it: I hope that there is some higher objective of your teaching and that you try to see the details of your everyday work in the light of that higher objective. When some item comes up for decision, your first question must be, of course: Does it serve some narrow objective prescribed by the needs of the community? If it does not, what is it good for? Does it serve your wider and higher objective?

30. It would help, I think, if we approached the coming discussions of this conference in the spirit of these questions.

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[1] Which is better called "heuristics" in other contexts: I used the latter term in my books:
How to Solve It, Princetown University Press, 1945; 2nd edition, Doubleday and Co., 1957.
Mathematics and Plausible Reasoning, vol. 1 and 2, Princetown University Press, 1954.
Mathematical Discovery, vol. 1 and 2, John Wiley and Sons, 1962/65.

CHAPTER IV

The Teaching of Mathematics at Primary Level

Lead paper by Miss E.E. Biggs, H.M. Inspector of Schools,
Department of Education and Science, London.

Introduction

** "In mathematics there's always a pattern –
you've only got to look for it"*

1. Mathematics is an abstract subject, and because of this, in the past, it induced fear in both young children and students. But in the last ten years there has been a fundamental change of outlook and the new generation is no longer condemned to a complete diet of instruction by teachers (however good). Nowadays, pupils are encouraged far more to think for themselves and they investigate mathematical problems in individual and sometimes highly original ways. The ethos of this new era is embodied in Professor Polya's words: "Abstractions are important; use all means to make them tangible. Nothing is too good or too bad, too poetical or too trivial to clarify your abstractions". The approach to mathematics, the way we present this subject to our pupils of all ages (including university students, as Professor Polya has so successfully shown us) is all-important. For this reason I shall consider approach first and leave content until later.

2. Here my main concern is with children between the ages of 5 and 12. For the more fortunate children of pre-school age, learning has proceeded naturally without effort or interruption. The newer methods in the infant school (5 to 7 years) begin in the same way with the direct impact of the environment and the child's own response to it. This permissive way of working, with children in small groups, was relatively easy to introduce in British schools since syllabuses are not externally imposed and head teachers can devise their own schemes of work and classroom methods.

Historical Background of Changes in Britain

3. The changes in classroom procedure in our infant schools began more than thirty years ago. Teachers concerned themselves first with the quality and scope of the materials they provided (often the normal materials of the environment such as clay, water, sand, scraps of textiles, leaves and shells) and secondly with the learning situations themselves. The quality of children's art and craft, movement and writing became more individual and imaginative; the narrow number syllabus was extended to include purposeful counting and measuring experiences. But until twenty years ago there was a marked contrast between the teaching methods of infant schools and

*See page 50

those of junior schools. At that time junior schools were influenced by examinations at the age of eleven for allocation to grammar (academic) schools. In these examinations the emphasis was on grammar and comprehension in English and on speed and accuracy of computation in arithmetic. It is not surprising that the first curricular experiments in junior schools were in art and craft and physical education, subjects which were not examined. However, in due course the emphasis in English, too, was gradually shifted from formal exercises to creative and imaginative writing with no loss of accuracy and much gain in fluency and vocabulary by the age of ten.

4. Experiments in the teaching of mathematics began about ten years ago. These changes were all the more far-reaching because they were brought about not as a result of external pressure from Universities or Education Authorities, but by the pioneer work of teachers in their own classrooms. They were part of a wider movement which had already affected many other aspects of the curriculum. Mathematics and science were no longer isolated; they now made a vital contribution to learning as a whole.

Discovery Learning in Mathematics

5. Let us take a closer look at "discovery" learning as this affects mathematics. "In some ways it resembles the best modern university practice".[1] I should be happy to think that most universities followed this practice! This is what Piaget has to say about it: "The goal in education is not to increase the amount of knowledge but to create opportunities for a child to invent and discover. Teaching means creating situations where structures can be discovered". When we say that initial curiosity is often stimulated by the environment the teacher provides, we are admitting that the teacher selects and structures the programme. Sometimes a worthwhile piece of mathematics will be initiated by a child with a special interest. More usually the environment and the teacher's questions will catch the child's imagination and sustain his enthusiasm. But the permissive classroom in which children are working in small groups, often on quite different problems, makes heavy demands on teachers and especially on inexperienced teachers. Both children and teachers have to accept considerable responsibility. Many teachers make a gradual transition. Most of the class work from a textbook while one group is provided with new materials; the teacher observes the children and asks questions. From month to month she increases the number of children working with materials until at last the whole class is involved.

6. What are the aims of "discovery" learning? First, to set children free to think for themselves. Secondly, to give them opportunities to discover the order and pattern which is the very essence of mathematics and which is to be found in the natural as well as in the man-made world. Thirdly, to give children the skills. Here I want to put a personal and perhaps extreme point of view. I believe that with experience teachers can plan each child's learning so that he discovers himself all the mathematics we want him to learn, even the skills. Our success in this will depend not only on the experiences we provide but also on the questions we ask. I also beleive that it is just as important for children to be encouraged to devise their own methods in

[1] "Children and their Primary Schools". H.M.S.O. 1967

written calculations as it is for them to make discoveries with shapes and patterns. Later in this paper you will see how children learn from one another to refine their methods.

7. I am convinced, from my own experience gained from working with children and teachers in many parts of the word, that the approach I am describing is not for "enrichment" only. It is an approach which can and should be used in every aspect of mathematics, at every stage and with children of all abilities. This does not mean that every child requires the same kind of experience – far from it. There are those who need little experience with real materials and who enjoy working abstract problems which develop from brief initial experiences. Other children require a lengthy period with real materials before they are able to abstract a single idea or concept. There are some children who will respond to open questions, and others who require more direction. Nor does discovery learning always require the use of concrete materials; it is the form of question we ask which determines whether we are challenging a child to think for himself – or telling him the answer. Dr. Schweitzer wrote: "Only those who respect the personality of others can be of real use to them". We must respect the independent personality of each child, and encourage him so that he takes the final step, however small, for himself. It is this independent discovery which gives him the satisfaction of success – and which requires all our skill to engineer. If we find ourselves telling the answer it is either because the child is not ready for the experience or because we have not planned the work to suit the ability of that particular child.

8. I should like to quote one example from an African school in Rhodesia because I believe that discovery methods would be as useful in developing countries as in Britain. Here I found that teaching methods were almost entirely instructional. I introduced simple materials, for example, congruent boxes, squared paper and string. At first children and teachers found it very strange that these should be used to make mathematical discoveries. Later, however, with adequate experience they became quick to see the possibilities, particularly the younger children.

9. Older students found no difficulty in making discoveries with number patterns – indeed, they had a strong sense of pattern. We had been working with the pattern of the 9 times tables. I wrote: 3, 12, 21, ... on the blackboard and asked them to continue the sequence and to construct similar sequences. They soon discovered that the final sum of the digits was always three. I then wrote several large numbers on the board and asked them to test these for divisibility by 9. "But, madam, they all 'refuse' ", they said with some excitement and were quick to discover why. I see no reason why developing countries should not be able to take advantage of these methods.

10. Because the form of the questions we ask – open-ended or directed – is so important, I should like to give you two contrasting examples:-

For several years I had given considerable direction to children and teachers in isolating the variables when working with a pendulum. For example, one assignment was: "Time the pendulum for 30 swings for lengths 6", 12", 18" as far as 48". Draw a graph. Can you find from your graph the length of a pendulum *which beats seconds?" A group of ten year olds performed the experiment carefully and obtained a reasonable result. A month later I discussed the pendulum with the same group. "What did you discover?", I asked. Their recollections were so hazy and confused that we had to start at the beginning once more.*

Last summer I was working with a group of nine and ten year old boys in downtown New York. We had decided to time various objects which the boys had chosen to roll down a long slope in the corridor. We had no stop watch, so I had added a length of fine string and a piece of plasticine to our collection. When the boys asked for a stop watch, I asked them if they could devise a means of timing from the materials I had provided. On seeing these Richard immediately suggested making a pendulum. It so happened that there was a large hook fixed at a height of 7 feet above the slope, to which the boys attached the longest pendulum they could make. "Where shall we start?", I asked. "Up at the ceiling, straight out", Earl replied. "Does it matter where we start?", I questioned. They decided that it did matter and we set the pendulum swinging. "Does it beat regularly?", I asked. "Let's count", they replied. But the boys decided that the pendulum was swinging too slowly for effective timing. "How shall we change the beat?", I asked. "Shorten the string", suggested Richard. "Lengthen the string", said Adrian. "Add more plasticine", said Robert. They experimented with different lengths until they were satisfied with the beat. "How shall we count?", I asked. "Number of swings in half a minute", said Mervyn. I gave him my watch with a second hand and he was soon timing confidently from any starting point of the second hand. "It swings much faster when we shorten the string", they commented as they continued their experiments with different lengths, "but it doesn't change when we alter the weight". These experiments were rough and ready and needed a careful follow-up with a good point of suspension, but I want to stress that <u>all</u> the thinking had been done by the boys themselves and the suggestions came from them and not from me. "The sense of personal discovery influenced the intensity of their experience and vividness of their memory".[1] *We were so engrossed that we forgot all about the purpose of the timing!*

But the initial impulse needs to arise from some experience in which the children are interested. *For example, the assignment: "Find the length and breadth of this room", may seem pointless to children, especially if they have done this already the previous year. But some nine year olds were enthusiastic when the question was worded: "Do you think this room is twice as long as it is wide? In how many ways can you find out without actually measuring?" One girl counted the bricks round the wall, a second counted the square tiles on the floor and a third counted the ceiling tiles. The others decided to pace the width and the length. They were at first puzzled and then amused when they discovered that each took a different number of paces. This investigation led to a discussion of 'normal' and 'average' pace, approximation and ratio. It also resulted in several oral calculations and was in all a most worthwhile investigation.*

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[1] op.cit.

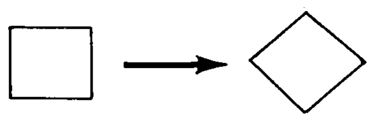
Discovery Learning in Arithmetic

11. I have said that it is important that children should be given the opportunity to devise their own methods of written calculation when they can no longer cope with the complexities in their heads. Let me explain how this can be done. First we must give young children (between the ages of 5 and 8) preliminary experiences which will enable them to learn what arithmetic is about. These experiences should include:

(a) Matching sets (one-one correspondence): setting a table for a meal, straws to milk containers, spoons to bowls, chairs to children.

(b) Counting, matching a number name to each object. Cardinal numbers. Numbers in sequence; the number line.

(c) Measuring; length, weight, capacity, time, area, using simple instruments. Learning to measure is a lengthy process involving several stages. For example, in considering length a child at one stage thinks that a pencil changes its length when it is placed in different positions. Eight and nine year olds who have not had the necessary experience say that a square geoboard turns into a diamond of different shape when it is rotated into a new position.



It is only by means of sufficient first hand experience that children come to understand that the length of lines does not change when their position is changed. In children's first experience of measuring they do not realise the need for equal units.

A class of eight year olds in Ontario had had their first experience of measuring. They told me excitedly, that the width of the room was 6½ bodies. "Whose body?", I asked. Three friends stood up, two of the same height and one, Roger, a head taller than the other two. "Would it have made a difference if you had used Roger only?", I questioned. "Yes, he would have got tired", was the amusing reply, which gave me the clue to their method. "Would your answer have been 6½ Rogers?", I asked. After some thought a girl said there would have been fewer Rogers, because he was taller. But the others maintained that the answer would be the same because it was the same room! It is only by means of extensive measuring experiences that children develop understanding and confidence in all aspects of measuring.

(d) Operations of addition, subtraction, multiplication and division with number, length, weight, capacity, money, time.

It is interesting to note that, in the real environment, subtraction and both aspects of division usually precede addition and multiplication.

For example, using two ribbons of different lengths, children first notice that one is longer than the other, and eventually may measure the difference (by complementary addition). One child asked, "How many ribbons of the short length can I cut from the long piece?" (the subtracting or grouping *aspect of division). Another child halved and quartered the long piece (the sharing aspect of division).*

12. What knowledge do children need of number relationships, including multiplication tables? If we believe that children should be able to perform reasonable calculations efficiently, they need to memorise (a) the addition and subtraction number trios, e.g. $3+5 = 8$, $5+3 = 8$, $8-5 = 3$, $8-3 = 5$; (b) the multiplication and division number trios, e.g., $6\times 8 = 48$, $8\times 6 = 48$, $48\div 6 = 8$, $48\div 8 = 6$. Once children have discovered the commutative laws of addition and multiplication, memorisation of these number relationships is halved. When they try to see whether subtraction and division are reversible (or commutative) they can be encouraged to extend their number knowledge to include the negative integers, using an extended number line, and fractions (or rational numbers). But we cannot escape this necessity for children to memorise the number relationships; first and foremost the addition and subtraction facts up to 20, then the extension to these facts to 100. For this extension a number line (100 units long) and strips (1 to 10 units long) are extremely helpful if children are to discover the repetitive patterns e.g., 9+7, 19+7, 89+7, and also 96–7, 86–7, etc. and also 96–9, 86–9, etc. These patterns must be known before children begin written calculations.

13. When children have this knowledge of number, what experience do we need to give them which will encourage them to devise their own methods of long multiplication and long division? Here, once more, are two contrasting examples:-

The first is of a group of 'disadvantaged' ten year old girls in California. They had estimated the number of pieces of cereal in a jar and were checking their estimates by filling identical mugs until the jar was empty. The first mug contained 59 pieces and there were 40 mugs in all. The girls had no idea how to perform the calculation, until one girl made an attempt to write forty 59s on the blackboard and add them up. The bell released her! Next day the teacher gave the girls six multiplication examples to do, one of which was 40×59. Each girl performed this successfully. These girls knew how to perform mechanical multiplication examples but had no idea of circumstances in which multiplication would arise.

My second example is of a seven year old class in Ontario. Their teacher had experimented for the first time in giving her class first hand experience with volume. They had made a most varied collection of containers. I picked up a bag of macaroni and asked them to guess the number of pieces in the bag. We all made wild guesses. I asked if anyone could suggest how we could check our guesses without counting every piece. "Count in twos or tens" was the first idea. A girl then said that if we could halve the bag, we could count one half and double the number, I showed the class a small cup and asked if this would help. Immediately I received two suggestions from boys. One said, "Fill the cup and count the pieces. Fill the cup until there is no more left and count the cups full". As soon as I left the teacher let the children try this suggestion. The cup held 110 pieces and there were 6½ cups. She wondered whether the children could solve the problem with this information since they had never tackled figures of this magnitude before. "Put two together and that makes 220", said one. "220, 440, 660", they counted. "I know half of 100 is 50 and half of 10 is 5, that makes 55. That's 715 altogether", said *another. The teacher was surprised at the confidence the children showed in attempting a problem entirely new to them. She realised that these children had encountered a situation in which multiplication was required. But they would need further experiments to prepare them for recording their own methods; gradually they would refine these as they had more experience.*

14. It is important to realise that mathematical symbols should be introduced *after* varied experiments. The sequence of events is: experience, discussion, recording in the child's own words, introduction of mathematical symbols as a shorthand form, written calculations devised by the child himself, (and gradually refined, with guidance from the teacher), practice as necessary. It has been interesting to notice that children who are being educated in this way require far less practice to maintain efficiency than children brought up on traditional methods. One period (40 minutes) a fortnight is often found to be sufficient.

15. Here is an example showing how children first devise and then refine methods of long division when the need arises:-

A group of eight year olds had been working with a calendar. How many weeks in 50 days? 80 days? 110 days? asked the teacher. One hundred and ten days drove them to paper and pencil. Each child produced a different method and, when all had finished, methods were compared. The most rudimentary method was:-

days			*days*	
110			*110*	
7	*1 week*	*The most efficient was*	*70*	*10 weeks*
103			*40*	
7	*1 week*		*35*	*5 weeks*
	etc.		*5 days*	*15 weeks*

All the children were attracted to the most efficient method and began to adopt it. One by one they refined the method still further, showing that they understood what they were doing and appreciated the need for efficiency.

16. I could quote many more examples of children's discoveries in measuring, fractions, and multibase arithmetic, but space prevents this. I should be delighted to expand these ideas in working sessions to anyone who is interested.

17. What is the place of commercial structural material in the learning of mathematics? I believe that children should first have extensive experience of the environment, but when, for example, they are beginning to organise their basic number facts, or at another stage their knowledge of fractions, the introduction of structural material will help them to do this. For example, when children can deal with simple fractions in practical situations and can find one quarter (and later three quarters) of varied materials such as a length of ribbon, a bowl of rice, a jug of water and a sum of money, they are ready to use structural material to summarise and gain further insight into their experiences. Later still they will meet fractions as rational numbers on the number line.

18. Learning by investigational methods has not led to any loss of efficiency in computation. The first schools in Britain to adopt the new methods were in urban

areas where classes were of over 40 and a selection test at 11+ was in operation. These schools kept careful records of the results at 11. It was found that in the first year there was no change. After this, year by year, the percentage of children qualifying for places at grammar schools increased until, in one school, it was 50 per cent. This local education authority has now abolished the 11+ examination! Of course the schools gave the children *some* written computational practice (one period every two weeks) but this was a considerable reduction on former methods, and yet the standard of computation had improved.

Mathematical Content

19. Mathematical content, has two aspects: mathematics in the classroom and the mathematical background which primary teachers require if they are to make the most of the situations which arise and are to give children the help they require.

20. There are certain topics which will arise in the classroom whether we plan these or not. I refer to statistics, three-dimensional and two-dimensional shapes, symmetry, similarity and limits. In children's attempts at communication they will use language (oral or written), tabular forms, diagrams, including mapping, pictorial representation of various kinds including block and column graphs, and line (relationship) graphs. The introduction of these topics requires and extensive knowledge of mathematics on the part of the teacher. I shall not attempt to develop each topic in detail; that is the task of the working parties. I shall quote one or two examples of children's work to illustrate children's difficulties and potentialities and to show how number relationships and spatial relationships are interrelated and reinforce each other:-

(a) Volume

The first is an investigation by some six year olds who had made a collection of small rectangular boxes. Their teacher asked if they could find the box which held most. After some discussion the children filled each box with sand and weighed box and sand. They set out the boxes in order but sand ran out at the corners, so they decided to fill the boxes with something else. This was not so easy and it was some time before a boy found that a cubical bead of a certain size would fit into all the boxes. The cubes were then taken out and arranged in a column above each box. At this stage no child noticed that the volumes could not be compared without counting because the starting points were different. Fortunately the classroom was overcrowded and a child bumped into the table and disarranged the cubes. This gave a boy an idea. He took a large sheet of ½" squared paper, arranged the boxes in order of volume, and coloured in a column of squares for each box, matching one square to each cube. This time he started, on his own initiative, from a common base line so that comparison was easy. "Can you see," he wrote, "that boxes 7 and 8 hold the same number of cubes? But they are very different shapes". This comment started a further investigation. Here we see the process of abstraction from boxes filled with sand (and weighed) to cubes, symbolising the volume, and finally to squares, by a matching process.

(b) Shapes

This example illustrates another important point. We live in a three-dimensional world, and two-dimensional shapes are abstractions from the real world. Therefore we should give children abundant experience with three-dimensional shapes before we expect them to abstract the properties of two-dimensional shapes such as squares, rectangles, triangles and circles. Before a child can recognise a square he needs to be able to sort a set of cubes from a set of cuboids and to put into words what he has done. There is no reason, of course, why we should not use two- and three-dimensional shapes simultaneously, as the following example shows:

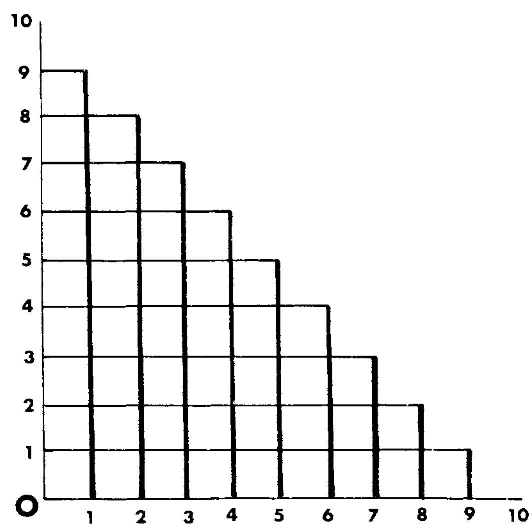
Some deaf five year olds had made a collection of containers of very varied shapes. Their teacher drew a square, a rectangle, and a circle on the floor and watched to see what the children would do. They sorted cubes into the square, cuboids into the rectangle, and cylinders into the circle. It took one girl some time to decide whether a packet of tea with two square ends should go inside the square or the rectangle. Eventually she decided on the rectangle because the packet had four rectangular faces and only two square faces.

(c) Transformations, Spatial and Number Patterns

Most of the spatial discoveries of young children are made as a result of movement and transformations. Pattern has a strong appeal to them.

I asked some nine and ten year olds in Philadelphia to make themselves a halo with a strip of coloured paper and then to experiment with this to find out whether they could change the area enclosed. They experimented with many shapes before they decided, by counting squares, that the circle was the largest shape. At the other extremity, a nine year old boy said he could squash out all the area! I then directed their attention to rectangles and asked them to draw me the complete sequence of rectangles whose perimeter was 20 units and whose sides were in whole units. They first drew these on the

FIG.1



floor in any order. I next asked them to cut the rectangles from 1" squared paper, to put these in order of width, and to mount the sequence on a piece of coloured paper. The paper was small so the children had to overlap the rectangles – and they were delighted to discover the "staircase" which a ten year old had predicted (fig.1)

I asked them to make tables of the width and length and area to see if they could discover why the rectangles formed a staircase. They soon discovered the pattern of the set of ordered pairs (W,L), W + L = 10, although they did not at first arrange these in order of width. When they came to make a table to show the patterns they decided that there must be a pattern in the area column because both the width and length columns showed definite patterns (fig.2).

FIG.2

W IN.	L IN.	A SQ. IN.	
0	10	0	
1	9	9	9
2	8	16	7
3	7	21	5

Excitement ran high when they discovered the odd number difference of the areas. I followed this by suggesting that the children cut out the sequence of squares with integral sides. The squares were mounted in various patterns. The perimeter and area patterns were recognised as soon as these were arranged in sequence (fig.3). Zero values were added later.

FIG.3

Edge	0	1	2	3	4
P	0	4	8	12	16
A	0	1	4	9	16

When, after some discussion, continuous graphs were drawn, the children's immediate reaction to the area/width graph of rectangles of constant perimeter and the area/width graph of the squares was to ask: "Where is the other half of the squares graph?" I suggested that they should count backwards and see.

Later I asked the children to reverse the constant-perimeter problem. "Fix the number of squares", they replied. I suggested that each child should choose a number of squares and make as many rectangles with these as possible. Once more they decided to use the square tiles on the floor and to *draw round these. But a boy who chose 5 squares found my suggestion too limiting and began experimenting on his own. He showed me the largest perimeter ("with squares as spread out as possible") (fig.4) and the smallest ("squares as close together as possible") (fig.5):-*

FIG.4

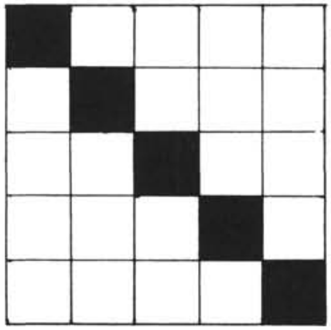


FIG.5

and then set out to "fill" the interval between 10 and 20 units with as many different perimeters as he could. We all became fascinated by this new problem and forgot our rectangles. Some time later I gave the boy's problem to a group of teachers in Ontario. Two teachers from secondary schools spent a morning on this problem, investigating all the possibilities, and discovered a group structure. They found this problem as intriguing as did the boy who invented it. This example emphasises another important point: open situations provide more possibilities for discovery than closed situations.

It was some time before we returned to our original problem: the pattern of rectangles of constant area (12 squares, 16 squares, 36 squares were eventually chosen by the children). This time the relationship of corresponding dimensions (3, 4), (2, 6), (12, 1) etc. was discovered before successive rectangles were cut out and arranged in a pattern. This constant product pattern $W \times L = 12$, and later the continuous graph, delighted the children because the graph was so different from the constant-sum graph.

(d) Symmetry

Many normal classroom activities (paper folding and cutting, painting and ink blots, tracings) lead children to make symmetrical shapes and to discover their properties but unless teachers know something of the mathematical significance of symmetry, they may miss opportunities for developing this. Children recognise symmetry at an early age.

Stephen, just five, was a very shy boy. His favourite occupation was painting. One day he dropped a splash of paint on his large sheet of paper just as he was about to paint a picture. He folded the paper to get it into the waste paper basket, and then it fell open. He was so excited by the pattern he saw that he took it to show his teacher. "Stephen's patterns" began to cover the walls of the school. Children's collections included leaves and flowers "like Stephen's pattern", and other flowers showing a different kind of symmetry.

A six year old had been struggling to fit the lid on a nearly square biscuit tin. As he came away his teacher asked him: "In how many ways can you fit *the lid on the biscuit box?" The boy turned to find out, and then, without further experience, replied, "Two ways it will, two ways it won't". That teacher realised the mathematical potential of the boy's struggles with the box and lid.*

A large class of ten year olds had been experimenting with regular symmetrical shapes. They discovered that the angles of a regular triangle were 60°; they calculated the angles of a hexagon (made with regular triangles) at 120°. "Three sides 60°, six sides 120°". The symmetry of this appealed to them. "Is this a pattern?", they asked. This sparked off an enquiry concerning the relationship between the number of sides a regular shape has, and its angle. They plotted a graph showing how the angle changed with the number of sides. The line joining the discrete points was in a curve (fig. 6), so the children decided to look for a turning point.

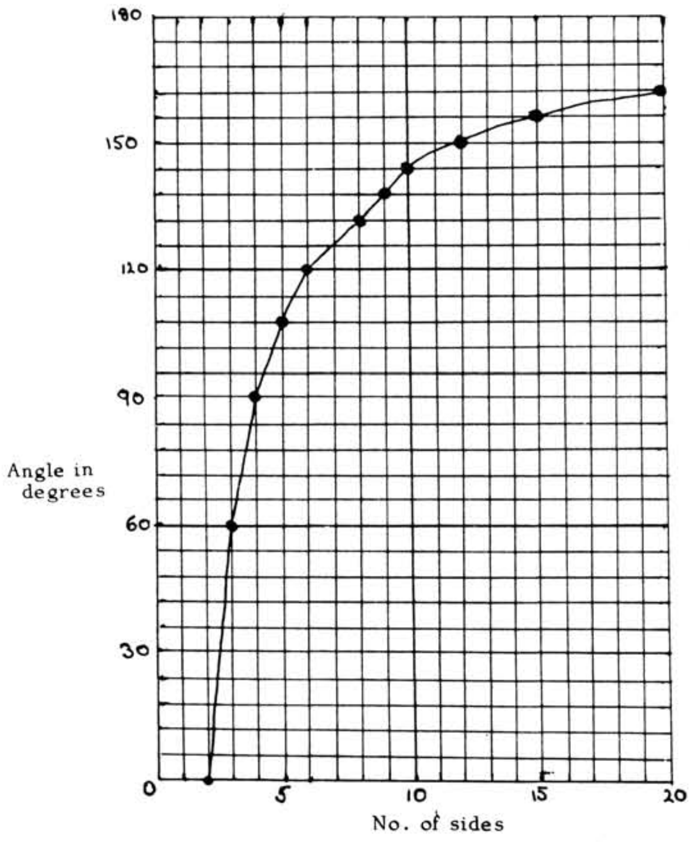


FIG.6

Their teacher was absent for several days, so they worked without interruption or supervision and plotted the graph to 240 sides. The line joining the points showed no sign of turning but they continued, calculating angles of sequences of regular polygons. The last entry was 450 million sides. "When we saw the pattern, we realised we never should reach 180° or turn the graph", they wrote. Throughout, these children were aware of pattern and symmetry in shape and in number.

(e) Similarity

I want to tell you one example of similarity from many, because this illustrates how easily mathematics can arise in other aspects of the curriculum.

A class of ten year olds who kept many classroom pets had noticed that the mouse seemed to feed for longer periods then any other animal. The teacher suggested that they should find out how much the mouse ate. So the children kept careful records daily for a month and discovered that their one-ounce mouse ate, on an average, half an ounce each day. Mark, with a baby brother of weight 8lb., found out that the baby had 6 feeds a day of 5 oz. "That's nearly one quarter of his weight," Mark wrote, He then kept a record of the food he ate himself. He found this to be about 1/25 of his weight. "Why do babies and mice, both small creatures, need more food?" the children asked. After much discussion with the teacher they decided that skin area and loss of heat might be part of the reason. The teacher (who had been doing some reading on the subject) suggested that the children should use a mathematical shape to find the relationship between skin area and weight (or volume). They chose inch cubes and used these to build a set of cubes of edges 1, 2, 3 etc. inches. They made a table showing the skin area/volume relationship. Here is the beginning:-

Edge (inches)	*Skin area (A sq. in.)*	*Volume (V cu.in.)*	$\frac{A}{V}$
1	*6*	*1*	*6*
2	*24*	*8*	*3*

This showed that the skin area/volume rate was halved when the edge was doubled, and a graph helped to clarify this still further. Some of the children found this idea difficult and repeated the experience using identical cuboids instead of unit cubes. Using the cubes the children discovered several other sequences and patterns concerning the perimeter and area of one face. From the number patterns they predicted the type of graph (straight line or curve). They found, too, that there were several other applications of these relationships in biology. (Geography provides many other important applications: scale maps and the globe, surveying, etc.)

(f) Limits

The last topic I want to consider is limits. I have already given many examples of this in the constant-perimeter and area section, and in symmetry. The children who continued their investigation to 450 million sides had first-hand experience of a mathematical limit when they realised that, even with that polygon, they would never reach 180° for the angle. But the concept of a limit can be experienced and understood by younger children.

I asked a group of 8 year olds in Ontario to make the largest square they could from a sheet of paper. After some false starts this was done and the extra piece removed. But in doing this the children had folded the square along one *diagonal and obtained a triangle. "What shape is it?", I asked. "A right angled triangle" they replied. "How do you know it is a right angle?" "It's the corner of a square", was the answer (showing that the children, too, understood the difference between intuition and mathematical proof). "Its two sides are equal", said Cathy. "How do you know?", I asked. "Because they match", "Because they are sides of a square", were the two answers given. In matching the equal sides the children had made another isosceles right angled triangle. "It's the same shape as the other". "It's half the other triangle and one quarter of the square" said another. With mounting excitement they continued to fold saying "$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ (words failed them after this). "What will happen eventually?", I asked. "You will get a very wee triangle but we shan't ever get to the centre", said Cathy. Scott remarked that as the triangle got smaller the paper got thicker until it was too thick to fold. At this stage I asked them to unfold their paper. Many comments were made on the pattern of triangles and squares. I asked them to cut the square in two along the diagonal and make the next larger triangle in the sequence. (This was very quickly done.) "Now join with a friend and make the next in the sequence, using four triangles". Before I could do this myself Janet shouted, "Me and Scott will do it and we can go on and on till we fill the room".*

This example exhibits several characteristics I want to emphasise. First, the children had an idea of a limiting sequence of numbers as well as of shapes. Also, the concepts of symmetry and similarity were introduced. But the material itself was simple. The work was not planned and I was carried on by the rising interest of the children.

In-Service Training

21. I have already referred to the need for teachers to know more mathematics, but important though this is, it is not enough. If teachers are to provide their children with opportunities for discovering mathematical ideas they must be convinced that this is possible from their own first hand experience. We have, therefore, developed our in-service training to provide teachers with opportunities for discovering by investigation the mathematics they need to know. They work in groups of eight, each with a tutor (often a leader, teacher or lecturer from a teachers' college). Each group contains infant, junior and secondary teachers. This contact is very helpful. There are various patterns of in-service training, for example, an initial 3 or 5 day course followed up at teachers' centres one session a week (afternoon or evening) for 8 to 16 sessions. It is important that most of the teachers' time is spent in working either at their own level, or in preparing work for the classroom. Often each group will cover the age range 5 to 16.

22. Teachers' centres were first set up in Britain early in 1963. These were provided by Education Authorities to give teachers opportunities of meeting in small groups at regular intervals to study mathematics and other subjects. Nearly 100 centres were set up in areas associated with the Nuffield Mathematics Teaching Project or the Junior Science Project, when these began in 1964/5. Today there are more than 300 centres; many of these now provide for the study of other subjects as well. Some of the centres are used as classrooms by day and as centres in the evening; others have been specially equipped.

23. Teachers from infant, junior and secondary schools frequently meet at these centres to study various topics in mathematics, to prepare work for the classroom and later to bring children's work to compare and discuss. In this way they are able to cover different aspects of mathematics and to learn from each other. Frequently the leaders are teachers although lecturers from colleges of education are usually called in to help.

Modern Mathematics

24. I have not mentioned so called "modern mathematics". The new ideas emphasise both the structure and the unity of mathematics and have helped us, too, to take a new look at more traditional topics in arithmetic, geometry and algebra. A knowledge of the foundations of number is of importance to all teachers. Many of the new ideas are already included in the curriculum because these are natural activities of young children, e.g., matching one-to-one, sorting (into sub sets), ordering relations. Communication of results has become surprisingly varied; language, mapping (a very useful way of recording), tabular forms, representations using three-dimensional material or pictures, and graphs of many kinds. These topics do not have to be specially introduced and given a new vocabulary and symbolism (except perhaps to use the word 'set' instead of group or collection). The formalisation of the new ideas is surely an activity which should occur at the secondary stage for most pupils but the concepts themselves, and a knowledge of them by teachers, are highly desirable if sound foundations are to be laid. In brief, a knowledge of modern mathematics is very useful for us but our children will come to no harm if they do not meet its more formal aspects until the secondary stage. It is essential that teachers should not be encouraged to experiment with new material *unless* they have sufficient knowledge and confidence.

25. In Britain the Nuffield Mathematics Teaching Project is experimenting with topics from modern mathematics in schools in more than a hundred areas. At their local centres teachers are asked to comment on their experiences of these topics.

Parents

26. But there are others concerned in new ways of learning, for example, parents and lecturers at colleges of education. The best way to initiate parents is to let them learn as we are encouraging their children to learn – actively, through real experience and by means of challenging questions. Many teachers' centres have fulfilled their function as parents' centres also.

Colleges of Education

27. It is fortunate that in Britain Her Majesty's Inspectors and lecturers from colleges have had annual joint conference on mathematics since 1956. We have worked together, too, on a number of teachers' courses. Many lecturers feel, as I do, that students, like fully fledged teachers, require to learn mathematics in the same active way as the children they teach. This is time-consuming and some principals are not yet aware of the time required to do this and to help to remedy deficiencies in the students' mathematical background. We still have a long way to go.

Conclusion

28. I want to end with my favourite story of children.

It concerns Peter, a ten year old, whose IQ was said to be 110 and who would not normally have been transferred to one of our grammar schools. Until he was ten Peter and some 40 others in his class had had a traditional education in arithmetic. In his last year in the junior school he was taught mathematics by a pioneer teacher. At first the children were hesitant and insecure. One day, however, Peter asked: "If I run a truck down the ramp and time it, then jack the ramp up twice as high, would the time be halved?" The teacher suggested that Peter and two friends should investigate the problem which occupied them for a whole morning. This was the first time Peter had taken the initiative. Some time later Peter brought the teacher his graph of how squares grow ($y = x^2$) and said he wanted to find two patterns. One was the pattern of areas under the curve at 1 unit intervals. The teacher was very surprised and asked, "But why do you think there is a pattern, Peter?" He replied, **"In mathematics there's always a pattern, you've only got to look for it".** *This started a six months' piece of work during which Peter, like Archimedes before him, discovered the calculus (both integral and differential). Peter had only one method at his disposal: he drew the curve ($y = x^2$) carefully, counted squares and approximated to find successive areas. At one stage he was faced by the sequence:*

$$\frac{1}{3}, 2\frac{2}{3}, 9, 21\frac{1}{3}.$$

This held him up until he decided to multiply by 3 – when he saw the pattern immediately. Peter's excited cry: "It's x cubed over 3", was reminiscent of Archimedes' "Eureka".

Peter's parents were totally unable to help him at home, where much of his work was done. Indeed, Peter developed too much independence to have allowed this. When his work was complete I talked to Peter about it. He had studied the sequence $y = x^2$, $y = x^3$, $y = x^4$. I asked him if his findings would apply if he continued the sequence in the other direction. He replied, "$y = x$, $y = 1$", then said, with mounting excitement, "I am going away to try this myself. I am afraid you will tell me", and he rushed out of the room until he had completed the problem. When he returned he said, "Smashing morning! It works that way too".

29. This story is significant for many reasons. Peter was not the most intelligent in his class, but once his imagination was fired he had the creativity and persistence to complete an outstanding piece of abstract mathematics. The problem was his own, therefore Peter was intensely interested in solving it, but he would not have asked the question had he not learnt mathematics in a creative, and permissive way. His first hand experience had made him aware of the order and pattern in mathematics. He possessed the skills he required to complete the solution. So Peter's work fulfils my three aims: releasing children to think, letting them discover the patterns in mathematics, giving them the skills. This way of learning mathematics is causing a revolution

in Britain, at all levels and with all abilities. We have a long way to go, but we shall not fail if we can convince all our teachers and equip them to work in this way.

REPORT OF WORKING GROUP A.1.

Chairman: Mrs. E.M. Williams (Britain)

Introduction

30. Group A.1 faced a double task in considering both method and content. To achieve both aims the group concentrated on the change of approach without specific regard to content and the introduction of new material, which would be taught by the newer methods. There was a ready acceptance of the change of approach from a predominance of class instruction to a proportion of active investigation by the children themselves. These changes are in line with modern thinking in education generally and their successful application in mathematics may lead to similar changes in other aspects of the curriculum.

31. There are major difficulties to be faced in adopting more modern methods. First there is a shortage of teachers adequately qualified for the new mathematics programmes; secondly there is a shortage of specially designed materials. Thirdly, where selection for secondary education depends on an examination the mathematics syllabus in primary schools may be restricted to the requirements of the examination. This report therefore presents a compromise between the desirable goal for the future and what is practicable at the present stage in different countries.

32. In defining the range of the enquiry the group analysed the limits of primary education in the areas represented. It was found that the majority primary education extended from 5/6 to 11/12 years of age. Some countries still have schools covering the age range 5 to 15. For the purposes of this report only the first six years of primary education will be considered.

Active Experience as the Basis for Learning Mathematics

33. In active learning the emphasis is shifted from instruction by teachers to investigation by children themselves. The teacher creates opportunities for children to invent and discover by providing materials for them to handle which he or she knows from personal experience will cause them to ask questions and to initiate their own explorations. But it is not always necessary to give children real materials; sometimes they will be stirred to ask about things they have observed and to pursue enquiries without prompting from the teacher. At other times their investigation of problems associated with shapes or numbers will lead them to invent a new problem. In brief, methods of active learning have caused a shift from over-emphasis on computation to a discovery of patterns and relations which will be found in experiences of space and number, in measurement of many kinds, in mechanical devices and in natural forms.

Arguments advanced in support of active learning

34. (a) Classroom experience in many parts of the world has indicated that children learning by means of their own explorations learn more

thoroughly. "Every teacher knows that a young child will learn more readily when he is engrossed in an activity". (Introduction to a West Indian Guide for teachers).

(b) Children of all ages and of all abilities from the least to the most able profit from these methods. Children who are slow at calculations may readily see spatial relations which will, in turn, help them to recognise number patterns.

(c) Children enjoy mathematics brought to them in this way and work with enthusiasm. The feeling of certainty that what they have discovered is right gives them confidence in their own powers.

(d) When these methods are applied to arithmetic and children are encouraged to devise their own methods for written calculations, they understand what they are doing and far less time is required for practice in isolation from experience.

(e) Since children are learning through questions and investigations the emphasis is on problem situations. The transition from the use of real materials to the consideration of verbal problems is an easy one because children have formed vivid mental images both of patterns they have encountered and of the ways in which they found or made them.

The teacher's role

35. Children need to handle real material; sensory experience is valuable in arousing their curiosity. The teacher selects material which has mathematical potential, observes the child's use of this and asks questions, when necessary, to help the child's thinking. Sometimes children start their own enquiries, at other times the teacher will need to ask an open-ended question or suggest an experiment to start them thinking. The form of the question asked is important. Open-ended questions stimulate a variety of suggestions. Directed questions rarely allow opportunity for inventiveness. (See The Teaching of Mathematics at Primary Level, Lead Paper by Miss E.E. Biggs, paragraph 10, for examples of directed and open-ended questions).

The value of discussion

36. When children are deeply involved in an investigation they will want to communicate their findings to their teacher and to their companions. Although there are various means of communication such as drawing, making models and writing, it is important that children should have opportunities to talk to other children as well as to their teachers either about their successes or about their perplexities. Such discussion may reveal language difficulties. This may be due to lack of opportunities for conversation in the home; or it may be that the medium of instruction is not the vernacular. If the mother tongue does not have the relevant mathematical phrase it may be necessary to provide the appropriate phrase in the ultimate language of instruction. For all children first hand experience and discussion will extend their vocabulary, enlarging the number of words they understand and words they can use spontaneously.

37. In some countries there is a tradition that children are not expected to speak unless they are spoken to; talking may even be regarded as a form of rudeness. In such circumstances free discussion will only occur if conversation is encouraged in the classroom. Since by discussion children clarify their thoughts and increase their command of language, such conversation is essential. This implies a freer atmosphere in the classroom than is found in formal class teaching.

The necessary materials

38. The materials required for an active programme are selected by teachers with the children's mathematical needs in mind. Through their choice the teachers are providing guide lines for their children's development. Some of the things needed will be found in the immediate environment and both children and teacher may be concerned in collecting these. But not all material can be provided in this way and it was strongly urged that Authorities should make money available for certain basic materials and equipment.

39. It is not likely that funds will be sufficient for all requirements. It was therefore suggested that teachers should be encouraged to make or improvise as many of the things they need as possible. If accommodation and facilities for this purpose are provided teachers will be able to share, possibly at a special centre, ideas about the use of local materials. Various ingenious schemes have been devised in several countries in Africa and elsewhere to produce equipment from local materials on a large scale.

40. Of course books are important. There are many books on the market for teachers and for children, including reference books of information. These can be a useful source of ideas for teachers as well as children. Each school should aim at making a varied collection using all available funds. Many teachers will require text-books to help them to provide the varied activities which children will need and also for practice in computation.

A suggested list of useful materials

41. *(a)* Shapes

Boxes, round tins and other containers; leaves, shells, fruits, seeds, flowers, inch and centimetre cubes, beads, balls, globe, mirrors.

(b) Sorting, matching and counting materials, structural materials

Seeds, stones, shells, small plastic toys, cubes, rods (length 1 to 10 units), number track, a number line painted on the wall or marked on the ground in a variety of relevant units, beads on a taut horizontal string.

(c) Measurement

Length : *Canes or softwood, string or fibre, homemade trundle wheel (yard or metre in circumference).*

Weight : *Balance scales, springs coiled wire (hair rollers, extension springs tendrils of climbing plants).*

Lever: *straight stick or rod to suspend, or to balance on a small wedge of wood, metal washers.*

Capacity: *Pots, gourds, cocoanut shells, pails, tins.*

Time: *String and bob for pendulum.*

Area : *Various materials to cover a plane (or curved) surface or a model, e.g. textiles, newsprint, leaves, seeds, squares, identical triangles, etc., mats, geo-boards or peg boards made of softwood with pins or pegs.*

Rotation : *compass, geared wheels, protractor, homemade clinometer.*

(d) Constructional Materials
Commercially produced strips and bolts or drinking straws, building blocks of various shapes and sizes (off-cuts), homemade level.

(e) Materials for Communication and Recording
Squared paper (in inch and centimetre uints), coloured paper and card, newsprint or other cheap paper, paints or dyes, brushes, coloured pencils or felt-tipped pens, abacus and rings of dough or paper beads, scissors.

Introducing Active Learning Situations in the Classroom

42. (1) The headteacher is a key person in the introduction of new ideas and could give a lead in his school. Exchange of ideas among the staff in the school is of first importance. To gain confidence teachers need to start with a small group of children. It would be ideal if the headteacher took the remainder of the class; alternatively these children could be set more formal work while the teacher was working with the selected group. The teacher has to convince himself that children using materials can discover mathematical patterns and relations for themselves without being given precise instruction. One method of organisation is to initiate a different group each day until teacher and children feel confident in this way of learning. The number of children included each day can then be extended.

(2) Teachers should be fully aware of the reasons for introducing new methods before they try these. They need to experience the active learning of mathematics themselves before they attempt this approach in their own classes. After teachers have attended an initial course which provides such opportunities, they need continued help while still experimenting in their classrooms. This help could come from leader-teachers, lecturers from teachers' colleges or advisory teachers. In some of the countries represented at the Conference full-time advisory teachers have been appointed to help in primary schools.

(3) Teachers embarking on a new syllabus find help in guides issued by education authorities and others. In some countries the syllabus and guides have been drawn up by groups of teachers, tried out in their classrooms, subsequently amended, and then approved for use. When introducing new methods it would be a great help if teachers could visit schools where good teachers were already using these methods.

(4) All those concerned with education should be made aware of the new methods: supervisors, headteachers, lecturers in teachers' colleges and parents. The

best way to help them is to organise meetings in which they are able to try for themselves, to learn new ideas by the new ways.

(5) When learning from experiences children cannot be kept in rigid rows. If classrooms are over-crowded children can work out-of-doors when conditions are suitable. For the future, movable flat-topped desks should be provided for older as well as younger grades. Extra tables, shelves, and a cupboard (steel where necessary) would be needed for display and storage. Additional display space is required for children's finished work. In planning new school buildings, Authorities should bear in mind that a classroom programme which involves considerable activity for the children requires different kinds of buildings and equipment.

Fundamental Content

43. Children's first experiences of mathematics should include opportunities for imaginative construction with objects which attract them. Through such activities they grow aware of the characteristics which things possess, such as colour, shape, heaviness and texture, and of likeness and difference in the things they are handling. Relevant vocabulary should develop easily. The range of activities offered at the early stages provides the basis of early ideas of number and spatial relations and the measurement of continuous quantities. They will also help a child to form mental pictures of his actions which will later make mental operations possible. The various aspects of mathematics should develop side by side although, for convenience, they are considered separately in subsequent paragraphs.

Foundations of Number

(1) The pre-counting stage

44. At the pre-counting stage many opportunities for the necessary sorting and matching experiences will arise naturally at home and can easily be contrived at school; for example: sorting bowls, pebbles or packages for colour, shape, size etc. Children usually notice that one of the sets has more things in it than another and the idea of comparing arises. Matching will also have arisen naturally at home and in school e.g. in setting out bowls for a family meal and in arranging mats or chairs for each child. Matching (one-to-one correspondence) can therefore be used to compare the sets. During these experiences the language which expresses the relation of inequality will normally arise before that of equality and will be used more frequently e.g. *more and fewer* before *as many as* (and later *more and less* before *as much as* when dealing with quantities). Other ideas of correspondence arise when talking about pairs of sandals and the people they belong to, and the children belonging to a family (many to one); a cat and her kittens or a hen and her chicks (one to many).

(2) Beginnings of counting

45. At this stage children will recognise the smaller numbers, perhaps 1 to 5, and will use their names, matching these number names to sets of real objects. Before children can really count objects they must have experience of order; for example: heavier and lighter, longer and shorter, arranging three children in order of height.

46. The idea of cardinal number can be extended beyond the easily recognised numbers when children can make a sequence of sets (beginning with one object),

each of them with one more member than the previous set. For example, a class of children arranged themselves in order according to the number of children in their families. Each child drew all the children in his family, naming each child and including himself. The families were arranged in sequence with equivalent sets (families with the same number of children) placed together. Children who can recognise equivalent sets must already have realised that the number of objects in a set does not depend on their arrangement.

47. Experiences of order also occur when children are comparing the contents (in spoonsful or cupsful) of a set of utensils, the weight of a set of stones (shown on a stretched spring or by using identical washers on balance scales), the lengths of their own feet (marked on a strip of paper), so that they form a sequence. After this children are soon able to count in spoonsful, washers or unit strips. Further experience should include keeping a record of the weight of a growing animal and the height of a growing plant. This, too, can be done by direct transfer of the daily height to a vertical strip.

(3) Sequences

48. The naming of positions in a sequence: first, second, third, etc., arises in dealing with quantities as well as with sets, e.g. order of entering or leaving the classroom, the order of children scoring points in a game.

49. In addition to arranging a set of objects in sequence, children should be asked to make a variety of patterns with them. They will then discover the patterns which are characteristic of any particular number, e.g. the rectangular pattern made by the even numbers or the pattern made by packing seven discs close together.

A sequence of sets with cardinal numbers 1,2,3,4, etc. is well illustrated by a child stepping along a line and marking his steps as he goes. Such movement can be shown on a graduated number line which will then be available to extend the child's counting.

(4) Operations

50. The handling of sets of objects, of continuous quantities (length, weight, capacity, time) and of money, also gives rise to operations with numbers. These will include matching and comparing (by subtraction and division); combining and separating will involve all four basic operations. These operations may arise in any order. For example, finding one-half (or one-quarter) of a yard of fibre illustrates the sharing or fractional aspects of division. Finding how many 9″ lengths can be cut from a yard of fibre involves the measuring or subtraction aspect of division. Both these experiences can be reversed by the teacher by asking the question: "What length of fibre do you need altogether to give a 9″ length to each of 4 children?"

51. At first, recording will be in pictures or in the child's own words orally or in writing. Later the teacher can introduce the shorthand mathematical way of recording, e.g. $9 + 5 = 14$ to match the experience of 5 more children joining a set of 9.

52. A number track and rods (1 to 10 units long) offer further opportunities for experience with the operations which will now be more closely concerned with the numbers themselves. Movement along the number line illustrates the operations of addition and subtraction, multiplication and division and their inter-relationships.

(5) Tabulation

53. The tabulation of results found from a number line (and from other experiences) gives an ordered mental picture which helps children to recall the pattern of a sequence and makes a contribution to the memorisation of number facts. A confident knowledge of such facts (addition and subtraction before multiplication and division) is essential before children can undertake the calculations which they need to write. By careful planning of experiences and questions a teacher can allow children to devise their own methods for written calculations, for example, in long multiplication and division.

54. The relations shown in the tabulation of a sequence, for example the set of multiples 0,3,6,9, etc., can be represented:

(1) by arranging unit cubes and rods;
(2) by a block graph;
(3) by a column graph.

Finally, the relationship can be extended and can be represented either by joining any pair of corresponding points on two parallel number lines or by a *continuous line* referred to two number lines at right angles.

(6) Laws of Arithmetic

55. Awareness of these laws may grow from examining tabulations. For instance, addition or multiplication facts can be organised as a square array or table (Fig.1) from which symmetry about the diagonal will show the commutative law of addition:

2 + 4 = 4 + 2 = (6)

3 + 5 = 5 + 3 = [8]

+	1	2	3	4	5	9
1	2	3	4	5	6	....
2	3	4	5	(6)	7	....
3	4	5	6	7	[8]	....
4	5	(6)	7	8	9	....
5	6	7	[8]	9	10	....
.	.	.	.	.	.	....
.	.	.	.	.	.	....
.	.	.	.	.	.	....
9	.	.	.	.	.	....

Fig. 1

56. The corresponding multiplication square (Fig. 2) will show the same law for the operation of multiplication. This brings out the rectangular pattern characteristic of a multiple.

2 × 3 = 3 × 2 = [6]

6 × 1 = 1 × 6 = [6]

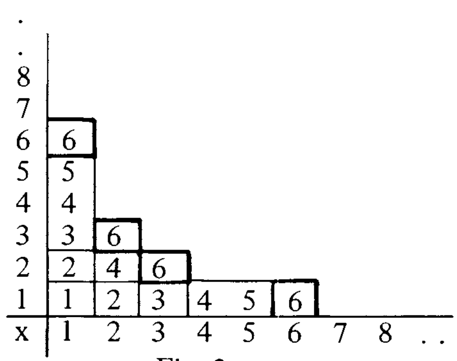


Fig. 2

(7) Systems of notation

57. Experience with other number bases helps children to understand the structure underlying the denary system as well. The

first encounter with such bases may be through packing a set of objects into square trays e.g. 5 by 5, rows of 5 and any odd ones. The use of unit cubes, rods and layers (as in multibase blocks) gives children one type of manipulative experience which leads into the more abstract representation on an abacus. The binary scale with its two digits, 0 and 1, has a special fascination for children (and adults); the fact that it is associated with the action of a computer gives a further attraction. Experience with this notation gives children a delighted appreciation of some of the number patterns which can be discovered.

58. The long sequence of powers of a small base such as four: 4, 4×4, $4 \times 4 \times 4$, $4 \times 4 \times 4 \times 4$,... may lead children to invent a shorthand notation related to the number of factors in each power. The conventional exponential form 4^1, 4^2, 4^3, 4^4,... will then be used.

Extensions to the use of numbers

(1) Fractions, decimals, ratio

59. The more experience children have of measuring, the more likely they are to be familiar with fractional parts of a unit. Measurement of the quantities mentioned earlier provide instances of "the bit over". The division of a quantity by a number such as 4 (finding a quarter of) leads rapidly to the idea of a fraction written as $\frac{1}{4}$. A wide variety of experiences of paper folding and cutting, of sharing a quantity or partitioning a set, is needed to establish the equivalence of fractions, a relation which is required for any calculations necessary. Children can devise their own methods for these calculations. The methods children invent are usually simpler than those we teach them.

60. Because we have to use two numbers to express a fraction other approaches have been tried out. Some teachers use the notation of fractions to mean the two operations multiplication and division; e.g. $\frac{3}{4}$ means divide by 4 and multiply by 3. Alternatively, if the two numbers are shown as co-ordinates on a graph we see the fraction represented by the ratio of the two numbers 3 and 4 or as the ratio of the co-ordinates (4,3). The ordered pair (4,3) is then used to represent the fraction $\frac{3}{4}$. (3,4) is sometimes written to represent the fraction $\frac{3}{4}$.

61. The equivalent value of 10 cents and a dime is a good introduction to both the idea of $\frac{1}{10}$ and to the decimal notation. 100 cents to a dollar extends the notation. The use of metric measures carries the notation further and it is hoped that countries which do not use the metric system will make plans for its adoption. The calculation and interpretation of percentages is easy when decimals are used. Integral percentages are hundredths and can be read immediately from the second decimal place. A representation on squared paper gives a useful ready reckoner for finding a percentage (e.g. 37/45) on two axes at right angles, provided the idea of ratio is understood.

62. One of the most valuable uses of fractions is to express a ratio, for example, the representative fraction used to show the scale of a map. Children from about 6 years of age will spontaneously use a rough scale in the models and drawings they make. When building two corresponding models using centimetre cubes for one and

inch cubes for the other, children will expect corresponding lengths in the two models to be in the same ratio and will verify this. (The volumes and areas of the two models will *not* be in the ratio of their lengths!). When accurate drawings are made the ratio of corresponding lengths can be expressed in fractional form. This gives rise to the word *rational* used to describe numbers expressed in this way. When rational numbers are represented on a number line children realise that any natural number can be divided by another number.

(2) Positive and negative numbers

63. When children try to see whether subtraction is commutative they sometimes think of extending the number line to the left of zero and so invent the negative numbers. The idea of positive and negative numbers may arise earlier through such practical experiences as reading temperatures above and below zero, finding the height above and the depth below water level, walking up and down stairs from the first floor or noticing the corresponding distances of reflected points from the axis of reflection.

(3) Statistics and Averages

64. Many enquiries initiated by children come from situations in and around the school, e.g. the number of children wearing certain types of clothing (blue shirts, etc.), the number of vehicles passing the school in a given interval of time, the size of shoe or length of foot. Usually the enquiries begin with a small group of children, then extend to the class and later may involve several classes and make the use of large numbers inevitable.

65. A block graph in which the children themselves are the units can be made, for example, when children choosing coloured rods from a container arrange themselves in rows according to the colour chosen. Representation of information collected in these enquiries should include:-

(a) block graph (in which the spaces are labelled). The number (frequency) of children in each set is marked on the vertical axis and heights of blocks can be compared.

(b) column graph (in which vertical lines replace the blocks, the points on the horizontal axis must be labelled).

(c) points marked at the tops of the columns. In a statistical enquiry involving a large number of children the points may be joined to show a trend. A clear distinction should be made between such a distribution graph and a continuous growth or relationship.

66. From these representations children usually comment on the most popular colour (called the *mode*) and on the range of the distribution. In some enquiries, for example into shoe sizes, the mode has relevance whereas the arithmetic mean (usually called the average) does not.

67. Where the arithmetic mean is relevant children can be asked to make a careful guess at this and to check their guess by calculating the deviations above and

below. Children sometimes use + and – spontaneously to denote quantities above and below both the estimated and the calculated mean.

Space and Shape

68. The study of movement, change and development is basic to the child's understanding of the real world around him. The transformation of the set of numbers (1,2,3,4,) into the set (3,4,5,6) by the operation of adding 2 is matched by the transformation of the points representing the numbers by a movement of 2 along a number line. The idea of transformation also gives an opening for imaginative pattern-making when a unit shape is translated along a line to make a border pattern.

69. Children will become familiar with certain basic two-dimensional shapes (e.g. squares, rectangles, circles, triangles) by handling a variety of three-dimensional objects which they should be encouraged to collect. By handling and comparing various shapes, children discover their distinctive properties, for example the likenesses of and differences between cubes and cuboids; squares, rectangles and rhombuses (diamonds); cylinders, spheres and cones.

70. The following activities give varied opportunities for investigation.

(1) Constructional activities

(a) building structures with unit cubes and unit cuboids,

(b) making patterns with unit squares, unit rectangles, unit triangles etc.

(c) making frameworks with split bamboo canes, drinking straws or meccano,

(d) model-making in card and other materials: sheds and buildings as well as the regular solids (tetrahedron, cube, etc.).

Interesting shapes such as bandstands, drums, balls, ice cream cones, the scoop for serving ice cream, pyramids could be made in the classroom and their properties investigated.

(e) making or investigating toys, for example kites, tops and hoops, doll-dressing. Mechanical toys and devices are not easily available in all countries, but some could be home-made, for example model aeroplanes, wheeled toys and an inclined plane (for experiment), go-carts, slings.

(f) Games such as cricket, football, basket ball and tennis can make children aware of the path of a moving object.

(g) A geo-board is useful for discovering relations of areas of different shapes.

(h) Paper folding, experiments with paint and mirrors lead to investigations of symmetry and enlargement (similarity). Children sometimes invent "co-ordinates" for themselves when they describe the position of buried treasure on an imaginary island they have drawn. When trying to find the treasure they realise the value of order in the pairs of distances used to fix the position of the treasure and ordered number pairs are then accepted. Co-ordinates can also be used in symmetrical reflections and other transformations.

(2) Mathematical ideas derived from these activities.

(a) The properties of various two-dimensional shapes, from making and handling frameworks with 3, 4, 5 etc. sides, using first equal then unequal strips; the rigidity of the triangle. Sequences of numbers from a series of frameworks of regular polygons made rigid by longer struts.

(b) Ideas of angles from a child's own observations of the hands of a clock and of other changes of direction. Angle properties of parallel lines and angle sum of a triangle from patterns made in paper folding. Rotation and gear ratio from cog wheels of different sizes.

(c) Mathematical similarity (enlargement of scale) in three dimensions, by building a sequence of cubes from unit cubes. Paths on the globe and on the earth. Similarity in two dimensions by building a sequence of squares from unit squares (and other shapes). Maps. Recognition that some shapes (e.g. cubes, spheres, squares) are always similar.

(d) Volume of cubes, spheres, cylinders, cones and pyramids from practical experience, for example, comparison of the weights of clay used to make the objects, or of the water contained by or displaced by them.

(e) Area: approach through irregular shapes, for example comparing the area of two leaves; counting the number of yam heaps covering each of two fields. Many activities lead to the understanding and application of area, such as gardening, weaving mats, doll dressing, making pictures with scraps of fabric. The making of nets of solids gives a useful link between two and three dimensions.

(f) The speeds of various moving objects compared with the child's own speeds.

(g) Transformation: The making of decorative patterns leads to an awareness of the various movements of a unit shape required to make different types of pattern: translation along a line, rotation about a point, reflection. Correspondences of points, and relations between lines and angles, can be investigated. The basic idea of congruence can be firmly established by means of these experiences.

Graphical representation

71. Representation on two number lines (axes) at right angles has already been mentioned. Certain types of relations occur frequently so that children come to associate the shape of a continuous graph with the corresponding number pattern. These include:

(1) Straight lines

(i) The pattern of the graphs of multiplication tables of 2, 3, 4, etc. as straight lines of increasing steepness. (Sometimes children ask if tables must stop at zero and so invent the negative numbers and extend the axes).

(ii) The perimeter/side graph for squares.

(iii) The circumference/diameter graph for circles.

(iv) A graph showing constant speed.

(v) The extension of a spring as weights are added to one end.

Children sometimes predict that the graph of the multiplication tables or perimeters of squares will be straight line because of the equal differences.

Edge of Squares	1		2		3		4		5
Perimeter	4		8		12		16		20
Differences		4		4		4		4	

(2) Graphs of squares and cubes

(i) The area/edge graph for the squares.

(ii) The area/diameter (or radius) relation for circles found approximately by counting squares. The number pattern of the area of squares is soon recognised.

Edge of Squares	0		1		2		3		4		5	-	-
Area of Squares	0		1		4		9		16		25	-	-
Differences		1		3		5		7		9			

Some children realise that because these differences are not equal, the graph will not be a straight line. Because the differences increase and form the odd number pattern they expect a rising curve.

(iii) The volume of cubes and of spheres will be investigated and will lead in a similar way to a curve.

(3) Constant product curve

(i) Examples of the constant product pairs have already arisen, e.g. in the multiplication square.

(ii) Dimensions of rectangles made with 24 squares. The complete set of rectangles with integral dimensions can be cut out and put in order: 1 by 24, 2 by 12, 3 by 8, 24 by 1. When children are asked to arrange these in order taking up the least possible space, they sometimes overlap them and then recognise the similarity to the pattern in the multiplication square.

Ordered pairs, (1,24), (2,12), ... , (24,1), with the relation $wl = 24$ give a continuous curve.

Links with other Subjects

72. Several very interesting projects were described by various delegates. Such projects gave children more confidence in their own powers particularly when the work was displayed. These topics normally arise in other aspects of the curriculum, e.g. social studies, science, art and crafts. Teachers with a sufficient knowledge of

mathematics can fully exploit the mathematical possibilities in activities of this kind. These should be carefully followed up and developed. The following examples were quoted;

(i) Social Studies: The banana industry had been studied in different ways according to the country of origin. Model making (sheds, trucks, boats). Different packing methods led to different developments in the classroom.

(ii) Art and Crafts: For example, construction of bird cages with palm pith; patterns designed for decorative purposes in craft of various kinds.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP A.1

Survey and Comments

73. The Chairman introduced the report with explanation of its considerable length. The early years were by far the most important, partly because many more children were in primary schools than in secondary schools and partly because the only education some thousands of children would receive would be a primary one. For this reason primary education was not only a preparation for secondary education; it must have its own particular goals. It was at the primary stage, also, that the greatest impact of the present changes would be felt. An emphasis on the development of children's thinking might have an appreciable influence on man's future power of decision-making.

74. The first part of the report deals with the changing approach. Then the content for the early years is discussed in some detail. Finally there is an outline of topics suitable for the later stages which will extend the range of mathematical thinking.

75. Subsequent discussion on the report centred on

(1) the problems arising when the mother tongue did not contain words appropriate to certain mathematical experiences

(2) the influence on the primary stage exerted by selection examinations for secondary education. These examinations aggravated the difficulties of adopting a fundamentally different mathematical programme since too much time was often devoted to preparation for such examinations.

76. Delegates expressed some anxiety lest the emphasis on children's practical experiences should lead some teachers to treat practical activities as ends in themselves. It was pointed out that the early sections of the report stressed the thinking that can emerge from children's individual experience and the need for teachers to guide their pupils – by questions, discussion and various forms of representation – to a clear awareness of the mathematical patterns that have been disclosed by their actions.

77. It was agreed at this plenary session that the editors should do some re-editing of the paper in the light of the discussion, and some modifications were made.

CHAPTER V

Mathematics in Secondary Schools

Lead paper by Professor W.W. Sawyer, M.A.(Camb.),
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University of Toronto, Canada.

1. Mathematics teachers are at present doing what philosophers urge -- asking what we do, why and whether we should. In an age of calculating machines need children know multiplication tables? Should we teach manipulation in algebra? If so, how much? Are logarithms obsolete? Should we teach trigonometry at all? If so, why? What parts of recent mathematics ought to come into the school syllabus? What should go out to make room for new topics? More fundamental, what considerations should determine our choice of syllabus?

2. Such questions deal with *what* is taught. Perhaps even more important is *how* we teach. Should mathematics fuse with science and the study of the environment, or should it be taught as abstractly as possible? Should it be formal or informal, rigorous or intuitive? In what proportions should discovery and "telling" be mixed?

3. A third type of question deals with how we achieve our objectives. It is easy to deliver a very inspiring sermon on the ideal school without explaining how to get there. In order to avoid a sense of unreality I will begin with some notes on the mechanics of educational advance.

4. I suppose that if you sent a watch to be repaired and it came back with the mainspring broken, whatever improvements the watchmaker might have made in the gear train you would feel he had done a bad job. But too often something of that kind happens in mathematical education. Most adults after leaving school not only know very little mathematics; they are incapable of thinking about it and afraid to try. Miss Biggs will surely stress in her talk that young children do not start out like that; they are interested in mathematics and willing to think about it. Accordingly, even for a child of rather low intelligence, the main difficulty in mathematics lies neither in the nature of the subject nor in the limitations of the learner, but rather in the attitude of the adults around him.

5. It is often thought that the solution lies with teacher training colleges. But this is somewhat naive. If a student comes to a college feeling critical of his own schooling, the college may be able to help him. But many students by 18 years of age have already a firm picture of what education is; it is what they had as pupils. They will return to school and teach as they were taught.

6. One of the most obvious (and least recognized) facts is the interlocking of the entire educational system. When universities criticize secondary schools they seem

only partly aware of the immense responsibility of universities for the state of secondary education. When secondary school teachers criticize elementary schools they do not always stress that elementary teachers are the products of secondary schools.

7. Mathematics is an activity and depends greatly on the attitude of the learner. Attitudes are formed young. The greatest influence on the future teacher is the elementary school; the next greatest, the secondary school.

8. One way in which elementary education concerns secondary teachers is that, in most countries, there is a movement of mathematical topics downwards. Secondary teachers complain they have not time to handle all the topics the university is handling over to them. The main reason is often a mathematical vacuum in the elementary schools. A.P. Rollett, in an article on English schools, mentioned abler children who did not start an intensive attack on algebra and geometry until they were 11 years old; he said they endured "mathematical stagnation". In North America, where a much milder exposure to mathematics used to begin at 14 years of age, these children would have been described as fiercely "accelerated". It is extremely desirable that a much richer content should be available to children 9-13 years of age, not merely to ease the secondary syllabus, but because these are the years in which the stimulus of new ideas most readily creates lasting intellectual interests. Secondary teachers should therefore be preparing for a steady transfer of serious algebra and geometry to elementary schools.

9. Secondary teachers will object that elementary teachers cannot handle such topics properly. But this ignores the fact that the future elementary teachers are to-day sitting in secondary classrooms. I would suggest as a *minimum programme* for secondary mathematics that every pupil should leave school with the basic equipment of a good elementary school mathematics teacher.

10. This means the following objectives, in the following order of importance. (1) The pupil should enjoy mathematics and not be afraid of thinking about it. (2) He should be able to grasp mathematical results informally, pictorially or in terms of a concrete situation. (3) He should have worked with mathematics in connection with simple scientific laws and with the environment generally. (4) In this way he should have acquired as much knowledge and understanding of arithmetic, algebra, geometry (and perhaps other mathematical subjects) as possible.

11. The application of this idea will naturally vary from country to country and from place to place. If elementary schools are already producing enthusiastic and active mathematicians, the secondary school should make sure it does not spoil what has been achieved, and should find it easy to build further. If children enter secondary school already disliking mathematics, a very radical rehabilitation procedure may be called for. If the secondary school was able to improve their attitude and give them a confident grasp of the topics mentioned in *Mathematics in Primary Schools** it could feel well pleased with itself.

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*Schools Council; Curriculum Bulletin No.1. H.M.S.O. 1965

12. Some secondary teachers have a very formal approach to mathematics and would be incapable of carrying out such a programme. The work should be done wherever there are teachers capable of it, and the authorities should emphasize that this is work of first importance, and the key to the whole national advance in mathematical education.

13. Mathematics involves general intelligence. The accomplishments of individual children will naturally vary. Complete success would mean that a child showed the same level of interest, confidence, initiative, originality and ingenuity in tackling mathematical and scientific problems as he did in any other department of work or play. Success would be judged not by multiple choice tests but by observing the child in a real situation.

14. Children, of course, would have to work at their own pace. I once ran a mathematics club in New Zealand where children could pass little tests rather like those for scout badges. Many of these were for *basic understanding of an idea,* e.g. "understands the use of x". In some American schools, each child has a folder which records his advance in reading. The same could be done with mathematics. This idea is elaborated somewhat in the Ontario report on Geometry, K-13.

15. This brings me to an administrative point. When I came to North America I saw something I had never seen before, a textbook labelled "Grade 9 Mathematics" – a scheme of work to be covered by all children in a certain type of class (e.g. academic, technical); a scheme laid down by the education authority and often enforced by inspectors. A teacher was thus under pressure to "cover the course", regardless of the fact that the slower pupils were bewildered and the quicker ones bored. Good teaching of mathematics in such an administrative setting is next to impossible. A real teacher is forced to become an underground worker.

16. Uniformity is particularly out of place when attempts are being made to change or to enlarge the syllabus. Inevitably, there will only be a limited number of schools where teachers are able to teach the new material effectively and inspiringly. These should not be held back until the whole city, province or country is ready – which it never will be. Rather, the strongest centres should go ahead and the influence allowed to spread gradually to others.

17. In a transitional period all kinds of informal measures may be necessary. A secondary school might (as has been tried in some places) offer a mathematics club, say once every 2 weeks, to the ablest and most interested pupils in the elementary schools near it. Secondary school pupils may meet two or three elementary school children and help them with algebra and geometry; in this way teaching talent may become recognized and an excellent training given to the future teacher. The ablest children in elementary schools should be encouraged to read ahead on their own. A survey should be made of what books children in fact can read successfully. On the analogy of music teachers, a good mathematics teacher might tour a number of elementary schools, giving one lesson a week in each to promote interest in the subject.

18. I have spent some time here discussing ages 9-13 because I am more and more convinced that this is the vital strategic area in which good teachers should be concentrated. The ages 0-8 are of course very important. However I am here discussing the role of the very best secondary teachers, those with a good knowledge of mathematics and the ability to bring out its interest and simplicity. I do not think they can do much by spending time with children below 9 years of age. The beginnings of arithmetic move very slowly. Also the work is simple; elementary school teachers can cope with it, once they have been persuaded that arithmetic is something you can experiment with and think about rather than learn purely by rote. On the other hand, if children from 9 to 13 are fully extended, many of them will get deep into fairly technical mathematics. Much of it they may read and do for themselves, but from time to time they will need advice which, in much of the world to-day, elementary teachers are not able to give. Occasional contact with the best secondary teachers will enable at any rate the strongest pupils to forge ahead. In the future some of them will become teachers and they will then regard it as normal to cover a substantial part of the present secondary curriculum (of many countries) in elementary school. In this way a richer syllabus will gradually spread and become established.

19. Incidentally, these ideas are not put forward as pure theory. My book, *Vision in Elementary Mathematics,* was based on experience with children in the age range 9-13. In my travels, I have found abundant evidence of children being able to cope with a far richer diet than anything that was being offered to them. For co-operation between secondary and elementary teachers, Barrie, Ontario, would be a good example of a place where this has been going on for several years.

Manipulation, Logarithms, Trigonometry

20. In an age of calculating devices the question arises at every level from arithmetic to calculus – how far need we teach only basic understanding and how far need we be concerned with slickness in manipulation?

21. The answer certainly lies somewhere between the extreme positions.

22. First, we should certainly get rid of the old idea that the use of any helpful device is cheating. When I see on examination papers, "Slide rules not permitted" I wonder if the next line will read, "All calculations to be in Roman numerals".

23. In general, we should be prepared to use, when appropriate, ready reckoners, slide rules, tables of integrals, graphs and any mechanical devices we can buy or make.

24. Some British examinations used to include very tricky and awkward integrals. I see little point in this. A course on calculus should cover the standard processes and the simpler results of integration. It should point out that certain expressions, such as $\sqrt{(1-x^4)}$ and $e-x^2$ do not have elementary integrals, and then set out the 4 main classes that are tractable – rational functions, rational functions of sin x and cos x, and so forth. The arrangement of tables of integrals should be explained.

25. Now, as to the need for some skill in manipulation. A new concept is often best introduced by some simple example involving a certain amount of calculation. In U.S.A., where even the teachers in secondary schools are very weak at algebraic manipulation, I sometimes tried to introduce calculus or matrices by a piece of algebra. Often I found the teachers' mental energy was completely absorbed in understanding the algebra, and they had none for understanding the new concept. Clearly this is an undesirable situation. The same applies at a lower level. Algebra itself can be interestingly introduced by studying some "coincidence" in arithmetic or through some simple scientific law. The pupils are expected to discover the regularity involved. Clearly this useful method of teaching is excluded if the children's arithmetic is too weak.

26. A calculating machine or a ready reckoner will give the answer to a specific problem but it does not help us to observe a law. It will give us the square of a particular number or find a particular product but it does not, when presented with a collection of figures, exclaim "Why, these are all squares" or "This is the 7 times table". Pupils need to become sufficiently familiar with multiplication tables, algebraic formulae and so forth to recognize situations in which these occur or to which they are relevant. Pattern recognition fortunately is an activity children enjoy and one that helps to fix the individual facts in the memory.

27. Even for problem solving certain material aids can be useful. For instance, to attack a problem in a traditional geometry course, a systematic approach is helped by a diagram showing the main theorems of Euclid. Generally pupils should be encouraged to make, and use, summaries of the information at their disposal. (Even many university students do not do this).

28. An approach that both saves time and creates interest is to use future theorems as exercises. (Theorems generally should be presented as problems, and not thought of as a separate category). For instance, in calculus to evaluate

$\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2\ldots$ is a reasonable exercise. But this integral is still meaningful when n is a fraction and allows us to define $n!$ for fractional n.

29. In trigonometry, deriving the formulas for sin 2A and cos 2A can be presented as a problem in co-ordinate geometry; if (c, s) is the point on the unit circle at angle A, what are the co-ordinates of the point at angle 2A?

30. Conversely, more advanced topics give a good excuse for taking another look at earlier questions and treating them more efficiently. It is really remarkable that simple formulas should exist for the sine and cosine of a sum; for most functions, f(A+B) is not at all simply related to f(A) and f(B). Matrices give one way of seeing why such formulas exist and deriving them. With calculus one can make plausible the connection of sine and cosine with e^{ix} and thus reduce all trigonometric identities to algebra.

31. There is the utmost diversity of views on the status of trigonometry. Some topologists, who never use it, regard it as a waste of time. At the other extreme,

most practical people still have to deal with solid objects that have definite shapes and sizes. In its traditional form trigonometry is still relevant for an architect or an astronomer. The trigonometrical functions are relevant to mechanical and electrical oscillations, to complex variable and to Fourier series.

32. I believe there is general agreement that work on the solution of triangles can be pruned drastically, particularly numerical work with logarithms and the related formulae. As a method of calculation, logarithms are nearly obsolete. (They may still be useful for finding a^n with large n). The logarithmic scale is still of significance and is readily introduced in connection with the slide rule. Incidentally the 17th century approach to logarithms was much simpler than the modern one, as logarithms preceded fractional indices by about half a century. There are considerable teaching advantages in defining logarithms first, and then defining x^k, for any k, as the number that occurs k times as far along the slide rule as x. Children are willing to concede that such a number exists, while they may have serious doubts whether $10^{0\cdot301}$ means anything at all.

33. I am inclined to regard algebra, co-ordinate geometry of 2 and 3 dimensions, and trigonometry as an indissoluble whole. Trigonometry enters naturally as the means whereby "a distance r at an angle θ" is translated into co-ordinate form. Co-ordinate geometry of 3 dimensions sounds very imposing, but in fact can be taught to young children of limited ability if it is presented concretely – with vertical straws on a perforated board, or, in an agricultural setting, sticks driven into a muddy piece of ground. Three-dimensional co-ordinates are an effective device for the design of any complicated solid object – an aeroplane, a building, a piece of metal work. At a more advanced level, co-ordinates provide a framework for most physical problems and also as a way of representing purely mathematical ideas; vectors, Hilbert space, etc.

"New" Mathematics: Errors and Possibilities

34. I do not myself accept any meaning of the "new mathematics" or "modern mathematics" other than the research advances since 1900. As commonly misapplied, these phrases are used to give prestige to any teaching innovation, good, bad or indifferent. In U.S.A. "traditional" is used to describe their tradition of very bad, rote teaching and thence (by double speak) to smear any topic previously taught. It is therefore necessary to discuss each innovation individually, and to appraise it as good or bad.

35. Ontario, lying uncomfortably close to U.S.A., has suffered on occasion from an uncritical acceptance of American ideas. An official curriculum introduced a few years ago organised the work of each year around such topics as the set of natural numbers, the set of integers, the set of rationals and so forth. This is an entirely incorrect basis on which to build, reflecting the procedure of the graduate school rather than the needs of the child. In the kind of approach visualized by Miss Biggs (whose prestige, I am glad to say, is steadily growing in Ontario) a young child may meet whole numbers and fractions in his first few encounters with measurement. The abstract approach has driven school mathematics and school science farther apart – the opposite of what is needed both on educational and on

technological grounds. The effects have been most disastrous for the less academic but more active and practically minded child. In a course for such pupils (15 years old) great stress was laid on "the set of irrational numbers". One over-zealous teacher had to be restrained from setting an examination question "Prove that the set of irrational numbers is not closed under addition". I cannot imagine how a teacher would hope to get technical students excited about such a topic which is completely irrelevant to their purposes and interests. What makes such regulations even more tragic is that all the relevant information about irrationals would arise, naturally and incidentally, as a passing comment on Pythagoras Theorem, which is of real and immediate concern to technical classes. The diagonal of a square involves $\sqrt{2}$. A teacher can point out that tables give approximate values of $\sqrt{2}$, but you will not get 2 if you square these, since in fact there is no fraction p/q whose square is exactly 2. The reason might be indicated; squaring causes each prime factor to occur an even number of times, a single factor 2 cannot be obtained.

36. Excessive logical analysis can inhibit mathematical thinking. This was demonstrated long ago on the largest possible scale, that of world history. The following passage is from Cajori, *A History of Mathematics*; the final sentence is quoted by Cajori from the mathematician Hankel; –

> "The Hindus never discerned the dividing line between numbers and magnitudes, set up by the Greeks, which, though the product of a scientific spirit, *greatly retarded the progress of mathematics.* They passed from magnitudes to numbers and from numbers to magnitudes without anticipating that gap which to a sharp discriminating mind exists between the continuous and the discontinuous. Yet by doing so the Indians greatly aided the general progress of mathematics. "Indeed, if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindustan are the real inventors of algebra" ".

The Indians, incidentally, were applied mathematicians. They wanted mathematics in order to do astronomy and (like the 17th century discoverers of calculus) did not have the inhibitions of the Greeks, who were not trying to go anywhere.

37. Good mathematics requires a balance between the Indian and the Greek approaches. Good teaching selects whatever mixture of these is most appropriate to the pupil being taught.

38. Now of course the American college professors who created the vogue for the set of irrationals have a perfectly sound point in logic. If for instance we determine $x = \cos 72^0$ by solving the 5th degree equation that expresses $\cos 5A = 1$, we are assuming that the usual algebraic procedures may be applied to the irrational number x. No doubt at some stage, for some students, the implications of this should be analyzed and, so far as possible, justified. What is overlooked is this. For a teacher in a neighbourhood of boisterous children, the first consideration is not the logical precision of his lessons; rather, it is the dramatic impact, how far the pupils will look forward to the lessons and feel that they are learning something exciting and worthwhile. If a course is dull, it does not matter how sound it is in

other respects; pupils will not pay much attention to it. In planning courses much more attention should be given to this question. At frequent intervals it should be clear to the pupil that the last chapter has enabled him to design, or make, or understand, or do something he could not do before. The first thing a teacher must prove to his class is that the course is not a waste of time, and the proof must be a spontaneous reaction in their hearts, not an argument grudgingly accepted by their heads.

39. In passing, I believe an excellent form of teacher training is to go into a street or park or public place where there are children over whom you have no disciplinary powers, and start doing something to see how many children come round you, how long they stay, and what questions they ask.

40. To return to the question of logic; a "preview" approach in mathematics is perfectly justifiable – that is, first to develop a subject informally, in a way that commands the intuitive assent of the learner, and to show what can be done with it; at a later stage, an analysis of the logical foundations may be undertaken. This is generally recognized (within the Commonwealth) in the teaching of calculus; an intuitive treatment should precede a course in analysis. In the same way, I would explain how Euler arrived at the connection between trigonometry and e^{ix} before discussing the Argand diagram or the logical account of complex numbers. First show that a method leads to interesting results, they tidy up the logic. For if it does not lead anywhere, why should the learner spend time on it? Of course it is good to prepare the ground for the later developments by indicating that loose use of infinite series can lead to fallacies, that pictures can deceive, and so on. How much this is done must depend on the teacher's estimate of the class he is teaching.

41. In recent years we have heard a lot about commutative, associative and distributive (C.A.D.) properties. It seems natural to ask – what is the role of these concepts in mathematics? – how did people come to think of them? These terms appeared first in the period 1800-1840, shortly after Gauss, Argand and Wessel had given a geometric interpretation of $\sqrt{-1}$. The entity i had always been something of a puzzle; it was not a number, and yet algebraically it seemed to behave like a number. When addition and multiplication of complex numbers were defined geometrically, the question naturally arose – what properties of these operations must we establish, in order to prove that ordinary algebra works for them? It emerged that most of the things done in algebra were logical consequences of the C.A.D. properties; any system with these properties could be handled *as if* it consisted of numbers. Hamilton, trying to generalize complex numbers, found quaternions with properties A. and D. but not C. Matrices followed soon after.

42. Great mystification is caused if teachers and pupils are told that some concept is important but are not shown significant applications of this. That certainly happened in the U.S.A. where words such as Set and Commutative received a kind of religious veneration. I therefore distinguish between the *unobtrusive mentioning of an idea* (which is an excellent way to prepare pupils for future work) and the *stressing of it,* which indicates that you are about to use it in deriving theorems of some substance.

43. The Americans were, I think, quite correct in holding that, from the very first lesson in arithmetic, we should be preparing children to do algebra one day. The early mention of the C.A.D. properties is in order. Thus we would not merely teach particular facts such as $2 + 3 = 5$ but raise questions of more general import, such as "When numbers are added, does the order matter?", "And for multiplication, does order matter?", "What is the craftiest way of working out $58 \times 3 + 58 \times 7$?". A good pupil in a traditional arithmetic class understood all these things; however it is quite sound to make sure they are brought to the attention of every pupil. The ideas involved are evidently helpful in arithmetic and in beginning algebra.

44. The C.A.D. properties can be used to introduce negative numbers, as in Durell, Palmer and Wright's book of 1920 and later in American S.M.S.G. schemes, though I would not regard it as good teaching to rely solely on this approach. Formal properties are singularly unreal to many children, and in fact do not clear up all the logical questions involved. Pictorial and inductive arguments should also be used to establish confidence in the use of negative numbers.

45. C.A.D. most properly takes the centre of the stage when pupils have met and used systems other than numbers (notably matrices and complex numbers) and have met new features, such as, in matrices, quadratics with an infinity of solutions. At this stage the question of which algebraic processes and theorems remain valid arises very naturally, and the justification of algebra by the C.A.D. properties makes sense to learners.

46. I have always thought that the splitting of the "in-any-order rule" into the pair of properties, commutative and associative, is a somewhat subtle affair, and some thought might be given as to how, and at what stage, this might best be done. The qualifications of the teachers are relevant. In some places, teachers are now making incorrect statements about sets where before they only made incorrect statements about numbers.

Modern Mathematics Proper

47. It seems reasonable to suppose that, as time passes, some mathematical results of the present century will find their way into schools. There is a difficulty in recognizing which results these should be. With classical mathematics, we have a fair idea of how each topic relates to other parts of mathematics and to applications. Recent mathematics is split into so many specialities and is so abstractly presented that it is often difficult to realize the inter-relations or to recognize that some paper is helpful for a problem one wants to solve. Associations and institutes should encourage mathematicians and users of mathematics to write understandable accounts of the origins and applications of recent mathematics. To some extent this is already happening.

48. A single example, taken from topology, will perhaps illustrate what I mean. It is a generalization of a classical result, in complex variable. Suppose we have a polynomial $f(z)$ and want to know how many solutions $f(z) = 0$ has inside a curve C. An interesting theorem gives the answer. Suppose that, as z goes round the curve C, the point $f(z)$ goes round the curve K, which, in our figure, makes 2 circuits of the origin. Then we can assert there are 2 zeros (or 1 double zero) of $f(z)$ inside C.

49. Now complex numbers are an extraordinary and unique system. Our theorem seems a very special and limited one. But in fact, by picking out its essential basis, it can be made very general. Suppose we consider not merely the contour C but imagine a membrane covering its interior. This membrane will transform to a membrane with boundary K, and there will be 2 layers of it over the origin. If, instead of requiring f to be a polynomial we simply require f to be continuous, the transformed membrane may have folds in it, but we can still assert there will be *at least* 2 sheets of the membrane over 0, hence 2 zeros of f inside C. This theorem generalizes to higher dimensions, and gives a useful way of locating the solutions of a complicated system of equations. Incidentally this method meets the criterion given by Professor A.J.M. Spencer in his article *The Education of Mathematicians for Industry* (Mathematical Gazette, October 1967); it allows us to find approximate solutions of real problems, rather than neat, exact solutions of unreal problems.

Mathematics and Utility

50. What mathematics do people actually use in life? In principle this ought to be one of the easiest questions to answer, for, unlike most educational questions it does not involve the nature of the human mind, about which we know almost nothing. It is relatively easy to count how many people do a particular job and what mathematics is, could or should be used in such work. Of course there is the difficulty of forecasting future developments. Yet at least we could establish short term trends – automation is wiping out the demand for a certain skill at so many thousand jobs a year, and is creating the demand for some other skill at such and such rate.

51. It is necessary to maintain a mind at once open and sceptical. An educational system is like an army; it takes time to move it from one spot to another, and still longer to bring it back if your first decision was wrong. I have been struck by the loose thinking about education of many, including mathematicians. Someone will say a certain topic is important when all he really knows is that it is much used in his own (non-applicable) research. Someone proposes a topic for the school syllabus on the grounds that it is used in electronic computing, but he omits to mention whether it arises in research, manufacture, maintenance or programming of computers. A mathematician in love with his speciality may claim it is practical because it has been used in a single scientific paper.

52. My other activities have never allowed me to give more than fragmentary attention to surveying the uses of mathematics. And in fact an adequate treatment

requires far more than can be given by one individual. I have suggested on various occasions that the users of mathematics ought to make a periodical report, available to teachers, on trends in the applications of mathematics. Even a small journal, abstracting the most important published information, would be helpful. The reports should make clear whether the developments affected industrial or agricultural countries, many or few workers, scientists or technicians, skilled or unskilled. Vague general statements would be forbidden, and concrete examples would be given of the problems involved. Such a periodical report would be of great value to many people besides teachers, and the Commonwealth probably has the resources to arrange it without setting up any elaborate new organisation.

53. Of course we cannot predict far ahead. Forty years ago (when teachers now retiring were just beginning) scoffing would have greeted a forecast that in 1968 unemployment would have been relieved because many people were gainfully employed trying to put a man on the moon. No doubt 2008 will prove equally unexpected. My proposal is not intended to provide teachers with a crystal ball to see the future; it simply tries to provide them with eyes to see what is happening now, so that we may make an intelligent guess what to expect to-morrow, and keep correcting our guesses at the first opportunity. Needless to say, schools should not try to teach the *details* of to-day's technology (which will soon be out of date) but *general principles* that are likely to endure. However to-day's examples, properly used, give reality and interest to lessons.

54. In 1945-7 the College of Technology in Leicester collected examples of the use of mathematics in the city. Since then of course computers and automation have brought many changes, but one conclusion still stands; the widest use of mathematics is not to solve problems, but as a language in which one learns science and technology. More recently, I made a small sampling of books on various subjects, to see the kind of mathematics used. Books are not always good indicators of new ideas (since authors and readers alike are usually unaware of recent mathematics) but they give some indication which traditional topics retain vitality. Elementary algebra is certainly one of these. It is hard to see how any development in higher mathematics can supersede the use of algebra for stating simple scientific laws, and making deductions by combining such statements. Fluency in reading algebra, the ability to appreciate the meaning of an equation or a graph and to associate it with its applications, is and will remain a most valuable asset, in everything from electronics to ecology.

55. In agricultural countries, interest may centre around the sciences related to biology; in these (as also in many industrial questions) statistics plays a large role. Accordingly, it seems that for biologists, and for others who do not intend to take very much mathematics, an attempt should be made to give familiarity with the binomial coefficients and their role in probability. In view of the normal error curve and the Poisson distribution (both of biological significance) enough calculus to understand e^x seems indicated. To treat e^x by algebra (as in Hall and Knight) is appalling. It is most undesirable to have people working with a symbol like e and having no idea of its meaning or derivation. In *Calculus Made Easy*, Sylvanus P. Thompson slipped e in very early and easily. It would be good if some such simple

treatment of a limited part of calculus could be taught as soon as a pupil had a grip on the basic ideas of algebra; calculus could then be used right through secondary school, for instance whenever a graph had to be sketched. Its ideas would thus become very familiar.

56. The ability to read science is of concern to all citizens, not merely to some employees. There was a scare recently about possible harmful effects of radiation from colour television sets. Here an everyday question involves two profound scientific topics – radiation and Mendelian genetics (probability and binomial coefficients again). In the newspapers there have been discussions of how many parts of sulphur dioxide per million parts of air industry can safely subject city dwellers to; whether the spread of industry will reduce the oxygen content of the air below that necessary to sustain life or, more conservatively, whether the increase of carbon dioxide in the atmosphere will cause the polar caps to melt and raise the sea-level by 180 feet; the poisoning of animals and human beings by the unwise use of pesticides; all kinds of unexpected and undesirable effects of substances sold in chemists' shops; and of course nuclear fusion. Not all the fears may be justified, but the very raising of these questions is a sympton of the enormous increase of man's power to interfere with nature. We are in the position of the sorcerer's apprentice; we have much more power than sense.

57. It may be thought that most people are incapable of appreciating the scientific issues that now permeate our lives. Indeed I do not believe that genetically mankind is any more intelligent than it was half a million years ago. But understanding is not a purely individual achievement. Literacy once meant the ability to master several thousand Chinese characters; to-day it means the ability to learn 26 letters and to cope with some oddities in English spelling, while in Ghana it simply means the power to master a fully phonetic alphabet. An ancient Greek needed something like genius to recognize that the earth was round; to-day a child, too young to go to school, may see on television a picture of the earth taken from space, and grow up never doubting that the earth is round. There are always ways to make ideas clear; our task is to find them. Earlier generations would have been astonished at the idea of every child learning to read. It may not be very long before we take for granted that every child can read mathematics.

REPORT OF WORKING GROUP A.2.

Chairman: B. Noonan, Ph.D. (Canada)

Introduction

58. Because of the great variety of forms of curricula throughout the Commonwealth, for teaching mathematics at secondary level, it was the opinion of this group that its work could best be carried out by initial discussion of detail which would lead to the enunciation of principles and recommendations which would be representative of the wealth of experience of attending delegates. Such principles and recommendations could then serve as guides, with all the reliability that experience, training and considered opinion can give.

59. Secondary education has different meanings in the various countries represented and the effect of, say, an external examination taken at the end of the third year cannot be ignored. Nevertheless, it was felt that the differences resulting from these variations were not so great as to inhibit a general discussion.

The transition from primary school

60. The first problem considered was that of transition from the primary school and delegates discussed the knowledge and skill which one would expect of entrants to the secondary school. It was agreed that:

> Any attempt to over-load the primary school syllabus should be avoided and that, although certain elementary skills were essential, the attitudes with which the child approaches mathematics were all-important. It was stressed that examinations should concern themselves not only with the acquisition of skills but also with the development of understanding and attitude, particularly in those countries where some selection for secondary education has to be made.

Planning a new syllabus

61. The provision of syllabuses in secondary schools was considered. The general view was that:

> It was desirable that teachers should have considerable freedom in deciding syllabus content, although it would be unrealistic to think that all teachers are yet ready for this responsibility. The setting up of Syllabus Committees in countries where they do not exist, should be encouraged. On these committees teachers should play a dominant role, but personnel from industry and Government should be included. In cases where external examinations are set for overseas students, the examining bodies should be guided by local requests.

Organisation

62. The group discussed school and classroom organisation and recommended that:

(a) School principals should arrange timetables to permit setting in mathematics. Under this arrangement a group of classes in a particular year have mathematics simultaneously, the students being assigned to classes according to ability in mathematics. This arrangement allows a student to transfer at any time during the year to a class which accommodates his ability.

(b) Classroom furniture should be chosen so as to facilitate group working and classroom activities which are inhibited by traditional patterns.

(c) Where possible, mathematics teachers should have individual classrooms to which classes may go in turn. Failing this, a minimum requirement is a mathematics room or a laboratory suitably equipped.

The common core

63. The desirability of providing a common core syllabus in the early years of secondary school was considered. It was agreed that:

All children in the first few years of their secondary education should pursue courses with a common core. Those children more able in mathematics should be given the opportunity to study topics in greater depth and to consider additional topics. At the other end of this ability range, topics should be treated in a more informal manner, but at the same time, such children should not be deprived of an enrichment programme.

64. The following principles for the design of a mathematics course were recommended:

(i) The establishment of a natural link with primary school teaching which should make the transition from primary to secondary school as smooth as possible.

(ii) In some situations a practical approach involving student activity would be a useful teaching technique to use. Teachers of mathematics in secondary schools should be familiar with and take an interest in primary school methods and encourage the continuance of these where appropriate.

(iii) In secondary school there should be a shift of emphasis to systematising and formalising, as well as expanding, the concepts and experience of the primary school.

(iv) To make the course relevant to the Community and its mathematical needs as far as they can be identified.

(v) The school curriculum should be integrated so that the contribution of a subject to each of its related subjects can be used to the best advantage e.g. the mathematical concepts necessary to physics should be co-ordinated with the physics programme and, on the other hand, aspects of physics which could serve to illuminate the need for mathematical concepts should be co-ordinated with the mathematics programme.

(vi) Proof or some other form of justification should be given for all formulae, algorithms and theorems considered.

(vii) In cases where the child's formal education is to terminate after 3 years, he must leave school not only with an acceptable degree of numeracy (i.e. insight into mathematical thinking in general and into all basic arithmetical techniques) but also with that flexibility of mind which would enable him to apply himself to problems he will meet as a member of his community.

The first three years

65. In considering the needs of pupils in their first three years of secondary education no attempt was made to lay down a syllabus or to suggest the order in which topics should be taught. The depth of treatment of these items, and the way in which they would be introduced would vary from country to country and indeed from class to class or even pupil to pupil. For example, although all pupils would be expected to be able to solve linear equations, quadratic equations would be included only when there was a clear need. In the opinion of the group, the following topics are those which should be encountered by all students.

Number

number line, extension to negative numbers, rationals, irrationals, and reals

place value, scientific notation (e.g. $2{\cdot}38 \times 10^{-6}$)

computational aids (e.g. the slide rule)

percentage, ratio, approximation and error estimation

Algebra

variable, functions

algebraic expressions and operations

relations, equivalence relations

inequalities, identities, formulae

equations, solution of equations

graphical representation

Geometry

area and volume, the theorem of Pythagoras

similarity, angle measure and angle properties,

elementary trigonometry

plans and elevations

co-ordinate geometry.

Identity elements and inverse elements should be studied in geometrical as well as in numerical and algebraic contexts.

Statistics and Probability

It was suggested that consideration of probability should spring from the pupil's own experiments, and that the statistics studied should be confined to representing data by means of graphs or tabulation, the interpretation of data exhibited in these forms, and the use of the measures, mean, mode and median.

66. Terms such as equivalence relation were not expected to be used by pupils, or necessarily to be defined explicitly, but nevertheless the ideas should permeate the teaching. For example, children should realise the similarity between 'is equal to' 'is parallel to' and 'was born in the same year as'.

67. It must be stressed that this skeletal plan must be supported by a full enrichment programme appropriate to the pupils concerned, for example, matrices might be studied not only for their own sake but also to increase the pupil's understanding of such items as operations, equations, inverse elements and co-ordinate geometry; while the study of transformation or vector geometry would, amongst other things, greatly increase a child's experience of spatial relations. The group felt that the choice of such topics should be left to individual countries or teachers.

Fourth and Fifth Years

68. When considering the syllabus for the fourth and fifth years, the group paid special attention to the fact that for many students this would mark the end of their formal education in mathematics. For such students topics with social and economic relevance should be given particular consideration. The following topics were thought to be appropriate:

(a) Linear programming
(b) Statistics up to sample theory and significance (including projects of an experimental nature)
(c) Co-ordinate geometry and vectors
(d) Operatives in systems other than the real numbers e.g. matrices, geometrical transformations
(e) Computing, flow charts, programming in some sub-sets of high level language, Computer appreciation
(f) Social arithmetic e.g. income-tax, hire-purchase etc.

Sixth and Seventh Years

69. The choice of topics at this level should be sufficiently wide to satisfy the needs of:

(i) those whose formal education in mathematics will terminate at the end of this stage;
(ii) future social and biological scientists;
(iii) future physical scientists and engineers;
(iv) future specialists in mathematics.

70. There was considerable discussion of the content of modern sixth-form syllabuses. It was agreed that some of the principles which should guide the choice of content are:

(i) the need to systematise parts of mathematics, for instance, co-ordinate geometry, properties of polynomials, elementary number theory;
(ii) the need for refining the ideas of proof and the use of axiomatic processes where possible. For example, elementary abstract algebra can be used to

justify rigorously results presented informally in the earlier grades. On the other hand, the introduction to the calculus should be based on an intuitive approach to limits;

(iii) the need to make mathematical models of physical situations: i.e. abstracting from a physical situation by symbolising quantities and relations, and arriving at mathematical conclusions in the model which can be interpreted in terms of the physical situation. Such illustrations are to be found in mechanics and probability;

(iv) the need to develop facility in the use of mathematical methods.

71. It is recommended that students should be familiar with some of the precise language of mathematics, including contemporary terminology relevant to the course they are pursuing.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP A.2

Survey and Comments

72. In the organisation of the Conference, the Planning Committee, when discussing secondary education experienced difficulty even when only some six countries were represented. In the range covered by the Conference, the difficulties were magnified in that there was a wider spread of aims, objectives and degree of selection for secondary education. In the United Kingdom, for example, there was secondary education for all pupils up to the age of fifteen years (soon to be sixteen years). Such a non-selective pattern did not apply to most of the developing countries, but some delegates reported experiences with various versions of secondary education such as vocational schools, secondary modern schools and community colleges for some pupils in addition to the strictly academic schools of the "grammar school" type. In most of the countries represented, secondary education was highly selective and was influenced by the requirements of a strictly academic School Certificate, GCE "O" or "A" level, as the case may be.

73. Evidence as to the climate of thought and consequent action was given by delegates from a number of the developing countries when they said:-

(i) "I think the kind of secondary education that is given in developing countries is always geared towards examinations" ... "If you look up the advertisements you will see that a School Certificate is always required and the children in the secondary schools are just aiming for this".

(ii) "The 'secondary modern' schools answered our problems in terms of keeping children off the streets, but then the children could not get jobs after completing the course, so these schools are no longer popular ... the grammar school still holds the supreme spot".

(iii) "... we have a new trade school for children who have ended their education at the age of fourteen. The objective of the trade school is to provide technical education to the age of sixteen years for the skilled labour my country needs. There will be a further development beyond the age of sixteen in a community college. What mathematics is to be taught at these two places is a big question".

(iv) "We have found that parents understand secondary education to mean the traditional form of education which leads to what is sometimes described as a "white collar job". There are not enough jobs of this kind; we need technicians ... and so our Minister directed that the basic education should be broader by the inclusion of such subjects as woodwork, metalwork, art and music".

74. This delegate then pointed out that such pupils often went on to a technical school for vocational training and they were often placed in jobs long before the end of their course.

75. In reply to a suggestion that any marked expansion of secondary education as currently practised in many countries would not of necessity be a good thing, it was thought that more secondary education of a different kind is necessary but that the whole problem is linked with the questions of teacher supply and training, of resources for education generally and the economic position of the country concerned.

76. The advice and recommendations of the report, being couched in rather general terms, places the duty of implementation with the country concerned according to its particular needs and problems. No specific attempt was made to lay down a working syllabus, or to order the items listed; still less to deal with detailed teaching techniques. The outcomes would be different in different countries. Such a possibility of variety was welcomed as being a desirable thing.

77. *A common core for the first three years of secondary education* received considerable support. Delegates felt that the common core outlined in the report could well serve the purposes both of the academic pupils and of the remainder who had not the same academic aspirations – and although it represented a desirable minimum it should be interpreted and enlarged by an enrichment programme designed to meet the needs of the pupils concerned. A note of caution, however, was sounded when it was pointed out that the crucial matter was the manner in which the topics were to be introduced, discussed and handled in the classroom. The existence of a list of desirable topics would not of itself ensure the interest and involvement of many of the pupils.

78. Any *common core,* since it represents the skeleton of a desirable minimum, can be criticised as unimaginative and lacking in bias towards the needs of any particular community. The point at issue is surely whether or not such a course leads the pupils to a sound grasp of the basic ideas and structures of mathematics; for it is on these that any vocationally biassed additions can be subsequently made. Furthermore, it is these basic ideas, provided that they have been suitably presented, which will enable the pupil to have a desire to continue learning mathematics and to find pleasure and profit in so doing.

79. Since in many of the developing countries, agriculture in one form or another was a staple industry, the question was raised as to the part a mathematical education could play in these circumstances. Whilst at present many of the agricultural workers were illiterate, or perhaps with a primary school education at most, this does not lessen the duty and responsibility of the authorities to provide these workers with a mathematical education. To quote a delegate "... even on a farm one must know some elementary mathematics, one has to make decisions, decisions for example as to what size of chickens to sell". The speaker claimed that a mathematical education contributed to this, and he advocated a continuance of the kind of approach given in the Primary School Report Group, as being suitable for such secondary pupils.

The limitations of a specific "agricultural mathematics" course at the secondary level were mentioned. Some countries had rejected such courses on the grounds that they did not provide for their growing needs and aspirations for wider technological development. Thoughts of such a restricted mathematical curriculum did not prove acceptable to the conference, and so further support was given to a "common core".

80. Since this Secondary Report suggested *forms of specialist mathematics rooms,* and the Primary Report (A.1) made no such suggestions, the distinction was questioned. It was pointed out that in a primary school the usual pattern was for one teacher to take one class for all the time in one place, and that in such circumstances it was easier to draw materials from a store and feed them, possibly by a trolley to the required place at the required time. In a secondary school staffed by specialist or semi-specialist mathematicians, the case for mathematics rooms or centres was considered a stronger one.

81. The recommendations of the Report relating to facilities for mathematics learning found general acceptance, as did the suggestions for suitable approaches to be used, and patterns of school organisation thought to be appropriate. Sporadic references continued to emphasise "practical approach", "continuing the experimental work begun in the primary schools" and the continuance of science and mathematics being studied together. The philosophy and the approaches found in the Primary School Report should extend to the secondary schools to a greater measure than at the present time.

82. The absence in the Report of any reference to the *history of mathematics* was deplored by one delegate. The meeting felt that a serious study of mathematical history was more appropriate to older students say at a Teacher Training College; nevertheless as one delegate said "perhaps every child should be made aware of how mathematics had developed and is developing: this is just a part of good teaching". Another delegate said that there was a danger of the history of mathematics deteriorating into a list of names and dates, yet it was important to include a well-devised course because many people appear to believe that the body of knowledge they acquire always existed and that they do not realise that there were times when it did not exist.

83. *Patterns of collaboration with industrialists and technologists* were discussed and served to highlight the necessity in some regions for forms of secondary education other than the strictly academic. In the words of a delegate "sitting around a table

with a number of industrialists writing up a school syllabus does not seem to be a practical proposition". Whilst this was agreed, it was felt that the users of mathematics had pertinent views to express, and members from widely different areas outlined patterns of operation which had been, or were to be, used. To take some examples:-

(i) In an African country a seminar lasting a week or so in which representatives of industry and of education agreed on desirable aims. Having established the objectives, it was left to the teachers to design a syllabus and introduce the content into the schools.

(ii) In a West Indian country a National Council exists for the control of technical examinations. This was stated to work fairly satisfactorily at the craftsman's level but that it was beset with problems at the technician's level, in that the courses at that level tended to correspond with GCE "O" and "A" levels. A student who successfully passed these, felt himself to be a potential university entrant, and so many of the students joined academic university courses and thus dropped a technological course.

(iii) Note was taken of the existence of an English project, *Mathematics in Education and Industry.* This project was based on the work of the Industrial Committee of the Mathematical Association: it seeks to effect liaison between industry and education and to develop syllabuses in the light of experiences gained. Revised GCE examination syllabuses have been prepared. Further information can be obtained from The Mathematical Association, 22 Bloomsbury Square, London, W.C.1.

For those wishing to make fuller study of vocational education a suitable source book would be:-

Vocational and Technical Education by Hugh Warren, Unesco (1967). This is a comparative study of present practice and future trends in ten countries (U.S.A., U.S.S.R., and eight European countries).

Another form of collaboration exemplified was where the representatives of industry came into the schools and colleges and became part time instructors in their particular specialism. In some regions, it was reported, the students go into industry for their practical training.

Vocational and technological education were important growing points for all countries, but no matter what form they take, they can only be based on sound mathematical learning in the primary and early secondary years.

84. *The relationship between the curriculum and the examination* was thought to be important. It was stressed that it was necessary that the examination should follow the curriculum and not, as so often happens at present, the curriculum be exclusively based on an examination syllabus. The setting up in England and Wales of the Schools Council for Curriculum and Examinations was instanced as a significant attempt to link these aspects of education. "It makes us, when we think about curricula, think about examinations and vice versa", said one delegate.

Delegates were very interested to hear of the objectives of CREDO (Curriculum Renewal and Educational Development Overseas), particularly that it sought to support rather than to initiate curriculum development.

85. The view was expressed that the rather cautious wording of the Report failed generally to communicate the thrill and sense of excitement of those teachers and pupils who are involved in new mathematical developments. This feeling of adventure was evident from many of the countries which had made a start on a new line. It was hoped that teachers and authorities would use the freedom offered by the report to experiment both with approach and with content, bearing in mind the needs of their country and the power of mathematics not only as an intellectual exercise, but also as an instrument for the general education of the person. In this respect it is hoped that the report will act as a broad guide. The content of the "common core", if treated suitably, could be regarded as the means of developing in pupils the power to discover and master problem situations together with the techniques and skills arising from them. At the same time it forms a broad basis on which further programmes can be built according to the needs of a particular country or of a particular situation. The Report should not then be taken as a policy of despair, but as an expression of optimistic hope capable of realisation.

CHAPTER VI

Assessment of Children's Progress, and Evaluation of Programmes: Purpose and Method

Lead paper by Professor J. Wrigley, B.Sc., M.Ed., Ph.D., F.B.Ps.S., University of Reading, England.

Introduction

1. A few years ago it was apparently easy to examine candidates in mathematics and it seemed comparatively simple to evaluate the mathematics curriculum. Twenty years ago at the primary school level the content of the teaching was confined to arithmetic and the debate was in terms of conventional testing of mechanical and problem arithmetic contrasted with objective, multiple choice tests. The latter could be demonstrated to be more consistent and reliable than the former but both kinds of examination in mathematics appeared to be more reliable than similar procedures in other subjects, particularly English. The situation at the secondary level was equally stable. There was an interesting and valuable move away from examining mathematics in separate compartments by means of papers in algebra, geometry, arithmetic, and trigonometry towards papers in mathematics as a whole. Even this highly desirable change took many years to come to full fruition in conservative England. At the eighteen year old level even less change took place until very recently.

2. The situation today in England is somewhat different and will become even more complex in the years ahead. Three developments have taken place recently, all in my judgement desirable, but all calculated to make the problem of examination, assessment, and evaluation more complex. The first is the development of new ways of examining which place much more emphasis on the assessment of course work and the opinion of the teacher. In England these developments have taken place around the new examination taken at the age of sixteen years, the Certificate of Secondary Education (C.S.E.). The change has come with regard to all subjects, but mathematics has been at the forefront in the various experiments which have taken place. The second is the development of new ways of learning and teaching which have come into existence in the primary schools and which, before long, will begin to spread upwards to the secondary schools. The trend is towards the abolition of rigid streaming by ability and the forming of classes of widely varying capability, particularly in mathematics. Efficient class teaching is almost impossible in such circumstances and the good teacher is forced to consider the formation of small groups within the larger class, and even the adoption of individualised learning by means of pieces of

work set for pupils to work on their own. Although I say that the teacher is forced to adopt these methods this does not mean that there are not good pedagogical and psychological reasons for adopting more active ways of learning. There are, and they are well demonstrated in Curriculum Bulletin No.1, 'Mathematics in Primary Schools' (1965 H.M.S.O.).

3. The third trend is the development of new content at both the primary and secondary levels. At the primary level we are no longer content to teach the old arithmetic; mathematics has taken its place. Sometimes this mathematics takes a quite traditional form, such as simple geometry and trigonometry, or the use of graphical methods; sometimes the change is more radical in terms of sets, transformation geometry etc. Particularly in our primary schools, the development of new mathematics and new teaching and learning procedures are often combined together in such a way that their effects cannot be separated. In the secondary schools, so far, the change has been, in the main but not entirely, towards new content. We have now, in England, well established new courses in mathematics, the Schools Mathematics Project, the Midlands Mathematics Experiment, the Mathematics in Education and Industry etc. A real breakthrough in examining came when the S.M.P. persuaded the G.C.E. examining boards to examine its work in its own terms. There was no questions of examining the pupils who had followed a new course by the examinations set for traditional courses. It seems obvious enough now but at the time it was a considerable step forward. More and more special courses, in many other subjects as well as in mathematics, are being developed and are being examined in their own right by special papers.

4. All these developments lead to considerable problems in examination and evaluation. Although the examination of candidates and the evaluation of curriculum are often considered as separate problems a little consideration readily shows that they are inextricably interwoven. An examination first differentiates between the candidates and then fixes a general standard of achievement. If we are examining pupils who have been taught to a new and radically different syllabus it is comparatively easy to grade the pupils in an order of merit (the first function of an examination) but it is much more difficult to make judgements about standards. Questions of comparability with the old standards arise; sometimes they can hardly be immediately answered. As soon as we make statements about standards we are also evaluating the curriculum as well as the teaching and the pupils. So our present task with regard to examination, assessment and evaluation is at once both more interesting and more difficult than it used to be. And in the immediate future the difficulties are likely to increase rather than be resolved.

Examinations

5. For a long time in the United Kingdom examinations of a conventional kind have played an important role in ensuring the maintenance and enhancement of standards and in providing an external measuring rod which could be seen to be fair to all candidates from whatever strata of society they came. The examinations at 11+, 16, and 18 have all played an important part in the democratisation of society and in making it possible for all children in our country to have opportunity for advancement. The fact that we now know that the opportunities have never been equal does

not alter the fact that external examinations have played an important part in encouraging social mobility. The trouble with most examinations is that we ask of them too much. We expect them to differentiate between candidates, to provide evidence of standards, to act as guides to good teaching, and to provide incentives for both pupils and teachers. We expect them to predict future performance as well as to certify that a candidate has completed satisfactorily a course of recognised study. Add to all this the fact that we examine candidates in their tens of thousands and it is not surprising that we run into difficulties. Ideally, we should separate these various functions, asking ourselves exactly what we aim to do, and design instruments to carry out our aims. But we are rarely able to do this.

6. The most interesting development in the realm of examining in recent years in England and Wales is the advent of the C.S.E. It was set up a few years ago to provide an examination suitable to the needs of pupils in our secondary schools who were not in the top ability groups. It was designed roughly for pupils between the 80th and 40th percentile of the ability grouping. Thus it was of a lower standard than the G.C.E. Ordinary level and intended to suit average children and above. It was to take the place of a number of external examinations which had been taken by these pupils and which had become rather remote from the needs of the pupils in school. To ensure that the new examination was relevant teachers were put in control of all the important committees and fourteen C.S.E. Boards were set up throughout the country. The philosophy of C.S.E. is that teachers and examiners should coincide, should often be the same people, that a teacher knows best the capability and calibre of his own pupils but that he needs the help and guidance of external examiners in the final determination of national standards. For many subjects, and for pupils of average ability it was clear that conventional methods of examination were no longer adequate so the C.S.E. was designed so that it could be taken in three different ways or modes. Mode I is a conventional external examination, Mode II is an examination set externally on a syllabus designed by the pupils own school and teacher, Mode III is an internal examination externally moderated. The latter is a means by which an enterprising teacher can examine his own work and his own syllabus in the way he thinks best within his own class, though of course he is subject to the checks of an external moderator. The C.S.E. Board has the final responsibility of making sure that work under Mode III is comparable in scope and in standard with work under either Mode I or Mode II.

7. The development of Mode III examinations with the associated trend to continuous assessment and the evaluation of course and project work has been uneven throughout the country and has depended upon the policy of the individual examining boards. Certain subjects of the curriculum such as rural studies, home economics, music seemed to cry out for the imaginative use of Mode III procedures, but a similar development has taken place in the more traditional and academic subjects, and mathematics has been one of these. In particular, those teachers who have been quick to see the value of new content in mathematics and/or the open-ended way of teaching have wanted to experiment with Mode III. The intention of many of these people has been to try to reproduce, albeit at a lower level, the kind of activity of a professional mathematician. This demands project work, essay-type questions, open-ended situations, and a whole new style of teaching, learning and

examining. The number of these *avant-garde* teachers is not necessarily large but it is growing. The problems raised for the moderator in such a situation are considerable. How is he to make sure that standards have been maintained, that the subject taught is still recognisably mathematics? My own view is that given tact and understanding a reasonable assessment and evaluation can take place. The teacher must be trusted to assess his own pupils to a large extent especially as regards an order of merit within the class. But the moderator might expect, even with the most esoteric subject matter and the most free and easy learning situations, that on some aspects of mathematics traditional questions could be asked about certain key aspects of mathematics. It is evident though that this kind of compromise does not completely solve the problem of comparability – the trustworthiness of the teacher and the wisdom of the moderator are essential elements in the situation.

8. I have been discussing rather extreme examples. The new ideas of teachers and examiners coming together, in one person, of ratification of progress rather than external examination, of continuous assessment rather than a once-and-for-all external examination, of the assessment of course and project work are present to a greater or lesser extent in all the three Modes of examination and in all the C.S.E. Boards. Similar procedures have been adopted in College of Education examinations for a long time. It remains to be seen whether a similar movement will take place in the universities and at the Advanced level of G.C.E. A system of internal examinations, continuous assessment, project work, open-book type examinations has not yet been extensively tried out in highly competitive situations. With most of these ways of examining the interaction between teacher and pupil becomes rather obvious. In examining a dissertation or a long essay the influence of the tutor on the work is considerable and it is sometimes difficult to know how to allow for this to give a 'fair' assessment. The fact that a similar situation exists in an external conventional examination (anyone who has examined large numbers of mathematics scripts at G.C.E. 'A' and 'O' levels will know that he is examining schools and teachers as much as candidates) does not mean that we have a solution to the problem.

9. The kind of development in examining that I have been trying to describe is comparatively sophisticated. It takes the risk of bringing in the teacher as an examiner, it admits openly the possibility of bias. It would perhaps have been an impossible development 50 years ago in the United Kingdom, it is still difficult to accomplish in a highly competitive situation. Such methods will flourish best in a system which provides expanding opportunities for education where the effects of failure can be retrieved at a later date. It is an interesting question as to whether a similar movement is suitable for under-developed and emerging countries. Since these new ways of examining (at their best) lead to more relevant and less artificial teaching and learning perhaps they are all the more essential. But they demand well qualified teachers who know exactly where they are going and what they are doing. The external London G.C.E. (and even degree examinations) set a standard for the whole Commonwealth, but sometimes at the expense of some unreality. Perhaps a compromise is necessary. Part of the examination in mathematics could be a conventional two or three hour written paper; this would take care of the essentials and provide an external yardstick. Part could be an assessment of course work, and open-ended examination – this would encourage real mathematics in our schools.

The Psychometric movement

10. Examinations and tests are both forms of assessment. Yet in England there has been until recently a sharp division between the two. Objective multiple choice tests have been viewed with suspicion by many of the people who operate our examination system. There are two main reasons for this suspicion – first that objective testing by means of one word answers cannot ever be an adequate form of assessment and second that even if the multiple choice activity were efficient as a test procedure the backwash effect on the teaching would be almost wholly bad. Memories of the coaching for the eleven plus examinations die hard in England. The first reason for suspicion is in my view not well founded. Good objective tests can examine even high level modes of thinking efficiently. The second reason is much more soundly based – until the competitive element in our examinations is reduced the backwash effect of any examination will be considerable – if we are to make it beneficial we may have to continue with the kind of interesting innovations I have described in the last section. The pressure of examining large numbers of pupils will force us in one of two directions – either towards more teacher and school-based assessments or towards more multiple choice tests which can be scored by machine. The examining boards in England are beginning to experiment with objective tests both in the C.S.E. and G.C.E. sectors.

11. Although my personal view is inclined towards the development of more school based examinations yet there are some interesting developments in the realm of multiple choice examinations. In mathematics one of the most interesting has been with regard to 'multi-facet' tests as opposed to multiple-choice ones. The idea is fully described in Examinations Bulletins No.2 and No.7 – The Certificate of Secondary Education: Experimental Examinations – Mathematics – *Mathematics (Bulletin No. 2)* H.M.S.O. 1964. *Mathematics 2 (Bulletin No. 7)* H.M.S.O. 1965. The normal multiple choice situation sets a candidate the task of selecting a correct answer or answers from among the alternatives offered. Such tests derive only one question from a given situation and therefore call for only one independent decision. It is however frequently possible to consider the same mathematical situation from a variety of facets and consequently to require from it a variety of decisions. In exploiting this multi-facet idea statements were phrased so that some were true and some were false and the candidates were required to decide the truth or falseness of each statement made about each situation.

Two examples will make the method clear:-

(1) If $a = \frac{1}{2}$, $b = \frac{2}{3}$ and $c = \frac{3}{4}$, then

(A) $abc = a^2$

(B) $a + b + c = \frac{6}{9}$

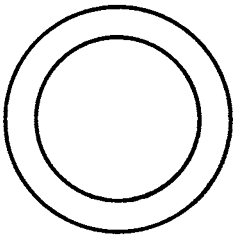
(C) $b - c = \frac{a}{6}$

(D) $2(c - a) = 3(b - a)$

(E) None of the above is true

	True	False
A	x	
B		x
C		x
D	x	
E		x

(2) C and S are concentric circles with radii 3 cm and 5 cm



(A) The area of C is $\frac{3}{5}$ of the area of S

(B) The circumference of S is $\frac{3}{5}$ of the circumference of C

(C) The circumference of C < twice the diameter of S

(D) All tangents to S are chords of C

(E) Any chord of S which is tangent to C is 8 cm long.

	True	False
A		x
B		x
C	x	
D		x
E	x	

12. The decision making situation combined with the obvious mathematical quality of the test described in the Examination Bulletins led to the method being adopted quite widely in England in the C.S.E. examinations. One of the criticisms of the method was that there was a guessing element involved in True-False answers. These criticisms were not entirely met by scoring the results of the test by means of a formula involving Right-Wrong answers. However the True/False element in the multi-facet situation is not essential – multi-facet questions could just as easily be phrased in such a way as to require the answers to be found and stated in a conventional way. For example (from Examinations Bulletin No.7):-

(1) *Situation* You are given the following information:

$$a = \frac{1}{6};\ b = 0{\cdot}60;\ c = \frac{2}{9};\ d = 0{\cdot}56;\ e = \frac{1}{4};\ f = 0{\cdot}64$$

(A) Place a, b, c, d, e, f, in order of size, the greatest first

(B) Find the value of (c – a) – (e – c)

(C) Is the average value of b, d, and f equal to, greater than or less than the average value of d and f?

(D) How great is the difference between $\frac{a}{c}$ and $\frac{d}{f}$?

(E) What is the value of the product abd?

Answer

13. Psychometric techniques have been developed for measuring the performance of items in tests. Good tests are tried out before actual use and item statistics calculated. Measures of item difficulty and item discrimination enable the test constructor to construct tests which are technically efficient in that they discriminate between candidates and they are reasonably reliable and consistent. Providing the test constructor also considers carefully his aims and objectives, prepares a blueprint

which makes sure that his aims are realised through an adequate sampling of the course of instruction, a thoroughly efficient test can be constructed. It is however very important to keep in the forefront of one's mind the aim of the test. If it is required to differentiate between candidates over a large range of ability then each item should discriminate – in a short test we cannot afford the luxury of items which all candidates either answer correctly or incorrectly. Hence the concern of the test constructors for item statistics. But not all tests, and certainly not all examinations, are of this kind. A teacher may need to use a test to see whether what he has taught has been understood and assimilated. In particular, there will be certain key things in any mathematics syllabus which must be understood and certain key operations which must be performed automatically and accurately. In the stress on teaching for understanding the need for automatic and accurate response is sometimes forgotten. In testing for such qualities we want a completely correct response from all our pupils. We are thus quite content to have items which do not discriminate and do not have any incorrect answers. In this situation, which ought to be common in teaching, item statistics are almost irrelevent. Similarly in curriculum evaluation we may be interested to know whether or not the children have understood key concepts. Here we may need items which measure understanding of these concepts. The items assembled for this purpose may be of the kind in which the response is either correct or incorrect and a whole test could be constructed so that something like a 90% correct response is required if the teaching and learning situation can be considered satisfactory. Similar considerations apply in programmed learning. The moral of all this is that the whole apparatus and theory of mental testing must be looked at carefully and the aims of the test or the examination must never be forgotten.

Curriculum Evaluation

14. Quite the most difficult task of those of us interested in assessment and evaluation is to evolve ways of evaluating new curricula. Whatever the problems involved in testing and examining children in traditional situations there is a vast store of sound theory and good practice which has been built up over the years. But the question of how to evaluate a new curriculum has hardly been tackled as yet. Indeed some of our most inspired curriculum developers seem unable to accept the need for evaluation except in a very rough and ready fashion. The pattern of curriculum development in both the United States and the United Kingdom over the past decade has been one of acts of faith and trial and error. This has been particularly so in mathematics and science. Men and women of ability and vision have become increasingly dissatisfied with the content of traditional mathematics both as regards its intrinsic mathematical value in this modern age and for its impact in terms of relevance, interest and difficulty on the pupils in our schools. They have had an unshakeable belief that new content should be introduced, and they have gone ahead and experimented with it. Sometimes the most interesting new ideas and new content have been suggested by professional mathematicians with little experience in the classroom. In Great Britain however we have avoided the unreality which might stem from a lack of practical experience of children by organising elaborate and extensive trials of the new materials and the new methods in school. We have brought in to the developments many teachers and many schools and made sure by feed-back

procedures that the lessons learned were used to modify the new materials. We have used a system of trial and error that is in itself a kind of curriculum evaluation which should not be despised or under-valued. Nevertheless we certainly need to go a good deal further. Curriculum development projects cost money and we need to know whether the money has been well spent and how to spend our money more efficiently on the next project. More importantly though, children's lives and experiences are affected by what they learn; how can we be sure that the new curricula are worthwhile? So far we have hired the best men available and given them trial and error apparatus to make sure that they can check their hunches and their practices. Can we do more?

15. The conventional answer to this question is as follows. The aims and objectives of a new development in curriculum should be set out, first in general and then in particular. The specific objectives which follow from the general aims should be able to be interpreted in terms of behaviour. "Down with non-behavioural objectives" is a slogan which has been seen on both sides of the Atlantic. Once the behavioural changes are known test situations can be devised to measure the change. Thus the success of the curriculum in terms of the developers own aims and objectives can be measured. It is then a separate issue, best done independently, to evaluate the intrinsic value of the original intention of the project as a whole. Such a programme for curriculum evaluation is logical – it might seem quite appropriate in a subject such as mathematics. It is however highly unpopular with many teachers and with many creative curriculum constructors. It has rarely been carried out in its entirety. The reason for the opposition is two-fold. First, the most creative minds simply do not work in this logical, analytical fashion. If other minds are set to work alongside the creative developers tensions immediately form. Second, the logical process of analysis seem to exclude a number of important intangible things which people feel to be important – the flair of the teacher, the hunch of the developer, the interest of the child – good teaching just does not work in the logical way suggested above.

16. In Britain the best we have been able to do so far is to develop a distinction between on-going and final evaluation. We are beginning to attach to any curriculum development team an individual whose role is to ask pertinent questions of his colleagues, to get them to clarify their aims and objectives, to organise tests of pupils understanding and of changes in behaviour, and to make sure that the feed-back of information derived from trials is efficient, adequate, and acted upon. Ideally such a man would be skilled in measurement techniques and sympathetic to the aims of the project. Ideally the tension which might develop between the evaluator and the developer would be healthy – this is most likely to occur when the individuals work closely together.

17. We have made less progress with large-scale independent evaluation. For the Nuffield Ordinary level G.C.E. Science development a research project has been set up to produce instruments to measure the knowledge and ability of pupils in science at the appropriate level. Such instruments, chiefly in the form of objective tests, have been designed to measure performance in science of pupils who have been subjected to both traditional and new science schemes. It is in this connection that

the different kinds of test construction mentioned previously can be employed. The tests will not be used to simply differentiate children's ability, nor will they be used to compare total score on the test between groups. What will be important is the pattern of scores obtained for differing sections of the test – both in terms of content and in terms of understanding of concepts. The instruments, when finally constructed, could be used as part of a national evaluation of science teaching in the country.

18. Two other developments are worth mentioning. For the Nuffield Primary Mathematics Project the assistance of the Institut des Sciences de l'Education at Geneva using Piaget-type test situations has been enlisted. A series of "Check Ups" have been designed specially for the project. The intention is to test the pupil individually in a reasonably well-structured and standardised situation to see if he has really understood a particular concept. The check-ups are not tests and there is no question of 'right' or 'wrong' marking. They are given to the children as and when the teacher feels the need to know how far their thinking has progressed. The methods follow the usual Piaget situation and contribute an interesting example of curriculum evaluation. Nevertheless they cannot be regarded as an independent final evaluation designed to see if the project is carrying out its aims and objectives. Rather they help the on-going evaluation and provide evidence of satisfactory progress on the part of the children.

19. Some people in Britain are worried in case the 'Check-ups', like examinations and tests before them, should be abused. One hears stories of the check-up situations being mistaken for the teaching and learning situation. The teacher could be content to teach simply for success in the check-up situation. But anything can be abused – the pressure on check-ups should surely not be particularly severe in a primary school situation in which the competitive element in mathematics is less than it has ever been before. And the procedure remains a most interesting and promising development in evaluating the success of teaching and learning in an intrinsic and fundamental way via the mastery of concepts.

Conclusion

20. The underlying theme of this paper is that examinations, tests, assessments, and the evaluation of curricula are all inter-related. In assessing and examining children we are to some extent evaluating our curricula, new and old, and we are also measuring our success as teachers. Traditional and modern mathematics exist side by side – indeed all sensible people are already synthesising the two. For our traditional procedures we had almost forgotten the need to evaluate their intrinsic worth, rather taking it for granted. We had simply examined children, relying on accumulated wisdom to set our standards. In the new situation the two tasks loom large on our horizon. Our problem is complicated by a double change – first in terms of new content and second in the development of different modes of learning. Examinations, both external and internal, still have an important part to play in the maintenance of standards, in the guidance of teachers, perhaps in acting as incentives for pupils, and for qualifying and selection purposes. The best examinations in the future will be a compromise between the 'teach and test' procedures which all good teachers need to employ, and the external checks which are still necessary. The new examinations

must not be allowed to inhibit the most enterprising teaching and they must allow intrinsic good teaching of all that is best in mathematics to take place. The best psychometric procedures should be adopted to make the examination as reliable and valid as possible.

21. Similarly, in terms of curriculum evaluation we are only at the beginning of inventing new procedures. Although we should strive to make our evaluations as scientific as possible we must also recognise the merit of rough and ready procedures. It may be that we could employ a whole series of measures to determine the success of our work, some of which at first sight have only a tangential relationship to our work. For example, measures of truancy, lateness, disobedience, proportion of children staying at school, increasing (or decreasing) numbers studying mathematics, threatening statements in the classroom, delinquency, numbers of books read etc. etc. All these could be quantified; taken together, with additions, the sum total could add up to a kind of evaluation.

22. The tension between examiners, testers, and evaluators on the one hand with inspired teachers and dedicated curriculum developers on the other is perhaps inevitable. What we need are inspired examiners and imaginative evaluators and such people are in short supply. The most hopeful long term solution to our problem is to make the teachers and the testers coincide in the same people, and to similarly combine the developers and the evaluators if not into the same people at least into the same team of people.

23. In mathematics, we have now had a good run for the development of new content and new ways of learning. Much remains to be done in terms of extending the ideas and the methods to more pupils and more teachers. And much remains in rationalising, analysing, and sifting the best in the new. In doing just this we examine, assess and evaluate. We must take care not to become arid in the process.

REPORT OF WORKING GROUP B.1

Chairman: Mr. Victor Matthews (Barbados)

Introduction

24. Teachers of mathematics are challenged to have higher aims. Refinement of aims affects the methods both of teaching and of assessment.

25. It is important that every "unit of teaching" should have an aim. It is appreciated that in a programme of experimentation the aims can be revised as the children discover unexpected patterns, relationships etc. However, children may be involved in a great deal of activity, yet it does not necessarily follow that they are learning any mathematics.

26. Whatever the programme, it is indeed important for the teacher to assess the progress of the pupils. The children's knowledge and skills must be tested from time to time. Their individual potential and attitudes should as far as possible be ascertained.

The Classroom

27. The teacher's checks or tests should help him to avoid wasting his own or the pupils' time. He must try to ensure that the child is ready to go on to the next step or to a new experience.

28. Frequent informal assessment on the part of the teacher may be one way of cutting down on too much formal assessment. However, from time to time the evaluation must be both searching and objective.

29. Opportunity should be provided for each pupil to measure up to something – the objective is for each pupil to achieve as much as he is able. One should take into account all aspects of the situation e.g. the total performance of the pupil in the school's overall curriculum.

30. A good teacher-pupil relationship is important in assessment so that the interest and progress of the individual pupil may not be impeded.

31. There should be a continuing evaluation of all pupils and programmes.

Tests and Checks

32. We should rely on the expertise and knowledge of the Piaget School as their ideas are relevant to the problem of trying to evaluate the level of a pupil's understanding. The checks used in the Nuffield Project are useful in this context. These will however have to be adapted for use in other countries. In particular they must be based on a good level of understanding of the language used between teacher and pupil. The introduction of Piaget-type testing should be gradual. Confidence in using these methods will vary according to the individual. In-service training of teachers should include testing procedures. The development of the use of such checks in a school could be helped by a teaming of teachers.

33. We have advocated a continuous evaluation of pupils. This evaluation will include verbal and written tests some of which may be competitive. However, a pupil is too often evaluated on a poor final paper his early enthusiasm or intermediate better performance gaining him no credit. Comments on course work should be recorded and not just the marks gained in tests. Work done on a unit or a project should be noted.

34. Easily tested topics or repetition of experiences which are easy to create lead to boredom. This could cause an apparent falling off in the child's performance.

35. Parents may be worried about the value of new approaches in the teaching of young children. The school, through such organisations as Parent-Teacher associations, should help parents to understand modern methods and the attitudes of new programmes. Experiences at home and school can be complementary.

Selection

36. Some form of selection of pupils on the basis of examination exists in all countries. Some countries have been able to postpone this type of selection until the

end of the Secondary School Course. In most developing countries, however there are limited facilities for secondary education. Appreciating the need to encourage social mobility these countries have to select pupils for secondary level, usually at 11+.

37. The type of examination used for selection varies according to the country's needs. By and large, however, they are externally set although there are examples in some countries of teachers' control in the design and content of these examinations. The prevalent tendency is to use standardised tests capable of being machine-scored.

38. There is no doubt that such examinations have considerable influence upon the teaching in the schools. It is most desirable that teachers and external examiners should work together in devising the content and method of such examinations.

39. It is desirable that teachers' opinion should be considered and that a verbal test suited to the particular country be used along with the mathematical test.

Purpose and Method

40. The Group considered the various purposes and methods of testing. Time was spent ranking the suitability of various types of tests for serving different objectives.

Specially considered were the following purposes:

Diagnosis, Selection and Prediction,
Motivation, Qualification and Speed

and the following types of test or assessment:

Multiple-Choice, Multi-Facet, Oral Formal,
Traditional, Practical, Open ended, Course Work,
Open Book, Teacher Opinion.

Examples of examinations which require tests for qualification are those for which a certificate is awarded (e.g. G.C.E.). The Group considered that types of testing preferred for selection were not the same as those preferred for 'motivating'.

41. There was consensus that:

Teachers should make use of a wide variety of types of tests or test items in assessing their students and that where teachers are able it is desirable that teacher opinion be given high rating in the external assessment of students for achievement or selection.

42. It was reiterated time and again that it was important to have *in-service courses* for teachers and that such courses should pay regard to problems of assessment. Enthusiasm alone is not enough.

Secondary Level

43. The types of tests used in the new programmes at Secondary level were considered and the trend to involve a multiple-choice test was noted.

44. It was felt that the multi-facet type of question would help to cover a good deal of ground and thinking but that ideally examinations should not be of one form. The majority felt that there must be a time limit on all written tests. The C.S.E. type

of examination in Britain was favourably considered and it was felt that teacher-involvement should be encouraged in examinations with similar objectives in other countries.

45. A delegate from one country described an examination system up to the Matriculation level in one State which depended almost entirely on internal assessment. This was regarded as highly desirable but it was recognised that for many countries it was as yet an impossible ideal. Throughout this country scholarships to enable pupils to complete their secondary education were awarded on the results of an examination involving a paper on Quantitative Thinking with consideration given to school assessment.

46. The involvement of Course Work in assessment was considered most appropriate in a stable situation.

Evaluation of New Programmes and Curricula

47. We are challenged to ensure that new curricula are satisfactory. The expense of development projects and more importantly welfare of the children involved must be considered. The group concerned with the assessment and evaluation of New Programmes submitted that 'New' and 'Old' Programmes cannot be easily compared. The objectives are different. Each programme can however be assessed in terms of its own specific objectives.

48. If teachers in the classrooms are involved in developing and planning the new programme, other teachers will be more confident in adopting it.

49. Teacher opinion is of prime importance. The need to evaluate continually and to modify where necessary cannot be too highly stressed. The opinions of teachers will need to be supplemented by tests and examinations from time to time. These tests must be carefully devised and it is necessary that teachers and evaluators should clearly understand how to proceed.

50. In certain projects central schools may be carefully chosen to take some of the tests. There are examples of this kind of evaluation on both sides of the Atlantic. Standards can be checked by a test including items based on mathematical content common to both programmes in conjunction with some kind of independent parameter such as an intelligence test or a paper on quantitive thinking. Checks made in project schools and control schools have also attempted to compare the level of pupils' interest and the attitudes of teachers.

Conclusion

51. It is important that an understanding of mathematical language be properly assessed especially with young children.

52. Continuous evaluation by the teacher is of the utmost importance and records should be kept. Not only will this help the teacher to assess the pupils but also to assess his own procedures.

53. The value of trying to ascertain the level of understanding of pupils cannot be underestimated. Piaget-type checks should be given to pupils when the teacher needs

to know how far their thinking has progressed. Other forms of evaluation, formal and informal, must still be applied. Traditional type tests with marking schemes awarding marks for method and accuracy are still considered appropriate for testing achievement although it is doubtful how they affect motivation. We recommend that teachers use a variety of tests in assessing their students.

54. Overall aims, as well as aims within stages of a programme, should be clearly stated and the programme or project can then be evaluated in terms of those objectives. A sharing of opinions about and comparison of results from similar programmes in different countries is to be encouraged. A cold assessment of new programmes must be done at some time.

55. In countries where a selective or an achievement test is necessary, such a test should include multiple-choice, multi-facet, and traditional types of items. Where possible, course work and practical tests should also be included. Where teachers are able, it is desirable that teacher opinion be given high rating in the assessment of students.

56. Where a qualifying or selective test has to be set externally we have recommended that teachers should be involved as much as is possible in devising the content and method of examination. It is encouraging to note that CREDO has offered to assist countries in problems of examination for selection.

57. Finally it is desirable that there be a further sharing of expertise and literature about assessment and evaluation between countries of the Commonwealth.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP B.1

Survey and Comments

58. Discussion centred around the statement in Paragraph 25 "It is important that every unit of teaching should have an aim". Some delegates questioned the desirability of this statement and were particularly anxious to know what was implied by the word 'unit'. It was accepted that in the planning of a series of lessons in mathematics aims and objectives should be both clear and explicit. But for an individual isolated lesson it was important to retain the freedom of the teacher to be adaptable and flexible. Delegates were reminded of the discussion on this matter in Professor Wrigley's lead paper where the point is made that good teachers do not necessarily work in a completely logical manner; they do not always proceed from aims and objectives to consequent behavioural changes and their subsequent evaluation.

59. It was felt by some delegates that the first statement in Paragraph 32 concerning the use of Piaget-type checks in the teaching of mathematics was rather strong. It was pointed out that the Geneva school of Psychology had its critics as well as its supporters. The general feeling of the delegates was perhaps represented by the later part of the paragraph. The use of Piaget-type testing in the Nuffield Mathematics Project had been a bold experiment, was still controversial, but the method had certainly proved useful. Such methods, if used in other situations and in other

countries, should certainly be introduced gradually, with caution, and with a full understanding by teachers which would probably only come with in-service training. Otherwise there would be many dangers.

60. There was an interesting and important discussion on Selection. It was first noted that in designing selection procedures those responsible should have clear objectives in mind when they are devising content and method. It was further felt that the phrase in Paragraph 38 "There is no doubt that such examinations have considerable influence upon the teaching in the schools" was an under-statement. Delegates were reminded of the discussion which followed the lead paper when in addition to the usual concepts of reliability and validity a new idea was introduced by the word *beneficence.* By this was meant all the possible good and bad effects of an extraneous nature which follow from any important examination. The backwash effects on teaching in schools which follow from a selection examination should never be ignored. As a result it is important to include questions of intrinsic mathematical value in the selection test.

61. There was some confusion as to what was meant in the report by a verbal test and the situation was clarified by a delegate from Trinidad. He pointed out that selection material in his country had been imported from Great Britain. Among the tests used was a verbal reasoning test. Psychologically this had been attacked because it was felt that the language of tests constructed and standardised in one country might not be suited to the culture pattern of another country. So the report meant that the verbal test (written not oral) should be one which was suited, not to the country in which it was manufactured and standardised but to the country in which it was to be administered.

62. An interesting discussion developed around the recommendation in Paragraph 46 that "ideally examinations should not be of one form" and in Paragraph 55 that tests "should include multiple-choice, multi-facet and traditional types of items". These recommendations had been made because the working group had recognised the importance of first deciding on the *function* of the test before deciding on the *type* of test. In some situations of a qualifying kind items and questions which could be answered by nearly all candidates were appropriate. But in highly selective situations where discrimination amongst the best candidates was needed then difficult items were needed. Considerations of beneficence and good backwash effects also suggested that a variety of forms of test were needed so that the teaching would not become stereotyped. These arguments were not accepted by all delegates in the plenary session. In particular the implied criticism of the possible over-use of multiple choice tests was not accepted by a Canadian delegate who said "There are those in my country who would strongly disagree with that statement and among those I would number myself".

CHAPTER VII

Teachers: Selection; Initial and Subsequent Training

Lead Paper by Mr. D.A. Perera, B.Sc., Dip.Ed., M.Ed.,
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Introductory comments including the delineation of the concept of training

1. The word "training" is used not only in such contexts as "teacher training" but also in other contexts such as "training of doctors and engineers" on the one hand and "training of semi-skilled craftmen" on the other. In many of these situations there is a period of training, very often in special institutions. A person completing a period of training successfully is believed to be capable of operating at higher levels of the cognitive, psychomotor and affective domains with an enhanced store of knowledge[1]. It is also believed that this development cannot be adequately and economically done otherwise. If it were not so there is no justification for special institutional arrangements providing periods of training. Hence "training" as used above, implies the existence, at any given point of time, of levels of achievement with respect to a body of knowledge that the trainee is to attain, over a period of time spent at an institute set up for the purpose.

2. 'Training' has another connotation as the following excerpt indicates.

> ...Training suggests the acquisition of appropriate habits of response in a limited situation. It lacks the wider cognitive implications of 'education'[2].

3. The phrase "teacher education" which is currently used seems to indicate that 'teaching' is not a routinised series of responses called forth by 'limited situations'. A teacher dealing with a set of pupils is in a complex and variable situation. But nevertheless there may be many common elements in a changing and complex situation as to warrant the assumption that some of the skills a teacher is to achieve are nothing more than the execution of "appropriate habits of response".

4. One method of examining the validity of the above assumption is to analyse the task or function of the teacher. This would also help in identifying the body of knowledge and the relevant levels of achievement that the novice is to attain.

A framework to view the main processes performed by the teacher in a cycle of teaching

5. In considering the task of a teacher one must consider the system in which the teacher operates. To fail to do so is to ignore a factor which exerts a very significant influence on the teacher. Hence a comprehensive framework should permit the

recognition of the fact that a teacher (of the kind that is the concern of this paper) is one element of an education system. Another factor that has to be considered is that a teacher, to perform his major task of teaching, may have to perform ancillary roles within the system which are different from his major role.

6. It is not claimed that the framework about to be described is either an ideal one or even an original one. The attempt is to combine one framework[3] which permits the task of the teacher to be viewed from a wide stand-point and another framework[4] which permits the teacher's major role to be analysed with greater specificity and concreteness.

7. The framework that is presented postulates that the personnel involved in an education system operates at different levels but that in each level the 'processes' which take place are essentially the same[5].

8. The different levels are named as follows:

Level I	:	Teacher – Pupil Level
Level II	:	The Principal – Teacher Level
Level III	:	The Field Supervisory Staff – Principal Level
Level IV	:	Directorate – Field Supervisory Staff Level
Level V	:	Ministry – Directorate Level[6]

9. It will be noticed that the personnel involved in the above scheme operate at two different levels except for 'Pupils' and 'Ministry'. The teacher's major role is at Level I. In Alles' words this is, "the basic level of operation, and the most fundamental one in relation to the educational system...All the levels justify their existence only in so far as they support and actively stimulate Level I[7] to operate with adequacy".

10. The leader at Level I is the teacher and his field of operation is the 'classroom'. The teacher participates at Level II but not as a leader. At Level II the teacher is but one element of a set of teachers which may include the head of the school as well. Therefore the teacher should not only be an effective leader at one level but also an effective participant at another. A conscious awareness of his role is more likely to lead to an adequate performance.

11. It will be observed that different social units are the concern at the different Levels. At Levels IV and V the thinking must be not in terms of the particular pupils that individual teachers have to think of, but in terms of much larger units such as a nation's children. At Levels I and II on the other hand a particular set of pupils, a particular set of teachers and a particular community are the main concern. An advantage of the framework that is being discussed is that it permits a consideration of the particular without losing sight of the larger groups.

12. It was stated earlier that at each level essentially the same 'processes' were performed. Alles presents them as follows:

"In this analysis it is postulated that:

(i) Notwithstanding the difference in the levels and the nature of the assignment, at each of these levels the major "unit processes" used to order means to ends remain essentially the same.

(ii) The major "unit process" that may be said to operate at each of these levels may be described briefly by the following terms:

1. Decision-making
2. Planning
3. Communication and Execution
4. Assistance, guidance and supervision
5. Evaluation and assessment

Ideally these major unit processes may be considered as quite distinct and separate; and may be assumed to occur cyclically. In practice this is not always the case and it is not always that they are followed in this sequence"[8]

In the remainder of this paper it will be assumed that the titles of the 'unit processes' are self-explanatory and hence no attempt will be made to describe them further.

Specific components of the teacher's task

13. The concept of a teacher performing the processes Decision-making; Planning; Communication and Execution; Assistance, Guidance and Supervision; Evaluation and Assessment would be more useful if we could give some answer to the question "Decision-making and planning about what?", other than to say, "Decision-making and planning about teaching".

14. Teaching is a term with a very wide connotation. The very environment is sometimes supposed 'to teach'. The concern of this section of the paper is with the teaching at Level I. But the relationship between a teacher and a pupil may be of many different kinds. To isolate the relationship that is under study the following definition advocated by Herbert is adopted.

"...An instructional relationship exists when teacher and student are in communication about subject matter"[9].

15. Though this definition limits the possible relationships it includes such instances as a student doing his assignments at home. In such a situation, the student and teacher are in communication about subject matter[10]. But it is generally regarded that in such situations, the teacher is not giving a lesson to the student. According to Herbert a further condition is necessary for an instructional relationship to become a lesson. The condition can be interpreted only in terms of what he calls the essential components of a lesson.

"... Six such components seem to make up the lesson: subject matter, the form of the subject matter, the form of the lesson, the media of the lesson, the grouping and location of students and teacher, and the influence techniques the teacher uses"[11].

The condition for an instructional relationship to become a lesson is stated as: *"A lesson occurs whenever there is an instructional relationship between students and a teacher who controls all six essential components"*[12].

Subject matter

16. Herbert makes a distinction between subject matter and subject. "The subject matter of a lesson is what is being taught through the giving of the lesson"[13]. What is being taught can relate to many different subjects. It is assumed that in planning a lesson, the teacher is taking steps to teach a small portion of a particular subject and for that purpose selects suitable subject-matter. In other words, it is assumed that a teacher planning a mathematics lesson has the intention of teaching mathematics although in the course of executing that plan, he may for example teach art, language, values, etc.

Subject matter form

17. "Subject matter form is the ordering or shaping of subject matter"[14]. It is the form in which the subject matter is presented to the pupils. Consider the subject matter topic 'the sum of the magnitudes of the interior angles of a triangle'. Some of the different forms in which it could be presented to students are as follows:

(a) State the property.
(b) Deduce the property using the chalk board to present the argument.
(c) Draw different triangles on the chalk board, measure the angles and state the property as a reasonable generalisation.
(d) With suitable questions and explanations get students to deduce the property.
(e) With suitable questions and explanations get students to obtain the property inductively.

The question can be for different purposes such as getting pupils to recall previous learnings, to refine a definition, to rephrase an answer, to predict, etc. Subject matter form is (along with some of the other components) a part of what is generally called the teaching method. It attempts to isolate, for analytical purposes, the character of that communication which the existence of an instructional relationship pre-supposes.

Lesson form

18. Lesson form is that component of a lesson which is concerned with the direction in which the communication of subject matter takes place. This too, is an aspect that would normally have been considered under teaching method. "Lesson form differs from subject matter form in that it relates to the ordering of the behaviour of those who are in communication, rather than to the ordering of the matter to be communicated"[15]. Herbert postulates three types of lesson forms at the first stage, each of which is further subdivided at two other stages resulting in seventeen different lesson forms. At the first stage the differentiation is in terms of the persons contributing the subject matter. All those lessons were the flow of subject matter is unidirectional from teacher to students belong to one category called Type 1. At the other extreme are the lessons (called Type 3) where the communication of subject matter is between students with no overt communication with the teacher, as for

example when a group of students discuss a suitable scale for a graph. Between these two extremes is the other category (called Type 2), where both student and teacher modify subject matter[16]. Within (say) a forty minute period there could be all three types of lesson forms repeated perhaps many times.

Media

19. These refer to the resources used to communicate. The voice is perhaps the medium most frequently used by teachers. Others such as books, charts, models, clay, projectors, chalk board, spring quickly to the mind. The bodily movements of teachers and pupils, the equipment used by a science teacher, the tests used by teachers are also used to communicate subject matter.

Grouping and location

20. "Location refers to the distribution and physical movement of teachers and pupils in the setting in which a lesson is carried on. Grouping refers to the putting into effect of criteria by which students of a lesson are classified for participation in the lesson"[17]. Teachers may not have a wide choice of 'the setting in which a lesson is carried on'. For most of the year the setting may be the same classroom. But within the physical space there are many possible choices for the location of teachers and students and the grouping of students.

Influence Techniques

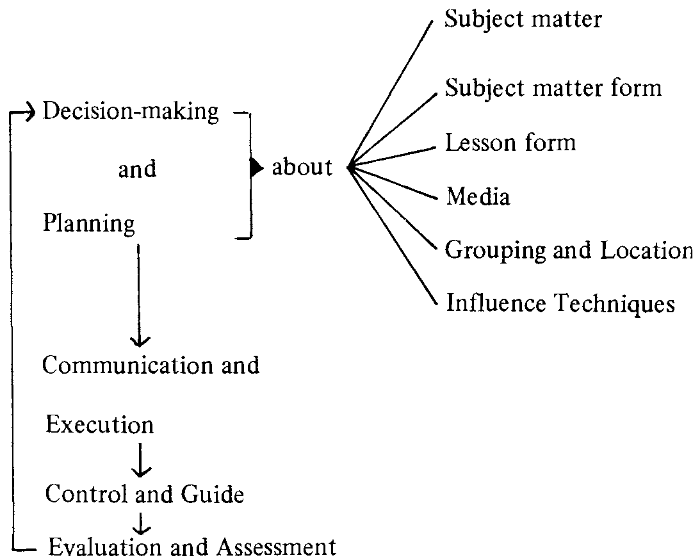
21. These are the techniques (part of teaching method) which enable a teacher to maintain his leadership role which implies maintaining control over the components of the lesson. A possible set of categories for influence techniques are "cues, selection of components, and techniques which are external to the components"[18]. Cues are the means by which the teacher lets the students know of impending changes in the flow of the lesson. A cue may be verbal such as "Can you remember...?" or it may be non-verbal such as a signal to get the textbooks opened. Herbert refers to cues as follows:

> "By such means students learn what is coming, what to expect, and what will be expected of them. In turn the students develop a language which gives feedback to the teacher. Together teacher and students create and maintain an orderly situation in which the teacher may control the components of the lesson"[19].

Control of the components of the lesson gives the teacher another set of influence techniques. New media may be introduced, the grouping may be altered and perhaps even the subject matter. The third category of influence techniques include such items as "threats, praise, promises, traditions, personal relations, competition, exhortation, class-room opinion, routines, rules of the classroom or school, and explicit commands"[20].

An examination of the teacher's task using the above framework with a view to obtaining some guide-lines for teacher education and selection

22. It is proposed to use the following frame-work for an analysis of a teacher's task.



Decision-Making and Planning

23. Decision-making implies the freedom to select one or more out of a given number of choices. It is worthwhile reflecting on the extent to which a teacher in an education system is free. By virtue of the fact that he is a teacher at a particular school many decisions are already made for him. In general, he will be teaching in an allotted curriculum area, to a given set of pupils, at and for predetermined times, within a given locality. In certain situations his decision-making sphere may be further circumscribed by the provision of approved syllabuses of instruction, course outlines, curriculum guides, textbooks, etc. Clearly the sphere of decision-making may vary from teacher to teacher in the same school, from school to school or from school system to school system. In the task of teacher education one aspect that has to be considered is how far the teachers are free to do decision-making.

24. How free should a teacher be to make decisions about subject matter? Is there such a thing as complete freedom to select subject matter? It will be recalled that the teacher is only one of the human elements involved in the education system. In particular there are others functioning at Levels II to V. All are engaged in executing, within the resources available to the system, the policy enunciated at Level V. If we believe with Peters that, " 'Education' involves essentially processes which intentionally transmit what is valuable..."[21] then it has to be conceded that there must be some concensus as to "what is valuable". Selecting subject matter is making a decision as to what is valuable. Hence if individual teachers are left completely free to choose the subject matter there may be little agreement as to what is valuable. National aspirations reflected in policy decisions at Level V may not be guiding, as they should, the actions of those at Level I.

25. On the other hand it will be agreed that complete restriction, in the sense of stipulating the subject matter for every lesson that the teacher has to give, is not practicable. Teachers have to be given some guide-lines as to what subject matter is appropriate. As stated earlier these guide-lines take a variety of forms such as syllabuses, textbooks, handbooks based on the texts, schemes of work for teachers, etc. In the Ceylon situation it has been decided that examination syllabuses are not specific-enough guide-lines even so far as subject matter is concerned. These syllabuses have been supplemented with what are called detailed schemes of work and further exemplified by pupil texts based on the schemes of work. It may be that for other countries the level of detail in the schemes of work may not be required. Whatever the level of detail that is regarded as being appropriate, one criterion the guide-lines should satisfy is that they should indicate to the teacher-education institutes, within the determined limits of practicability and desirability, what subject matter has to be handled by teachers once they complete their initial training. This not only does not imply that the subject matter is unvarying but implies a continuing effort being made to examine its suitability.

26. The subject matter identified by the guide-lines is one of the factors to be considered in selecting novices. Is the subject matter such that novices require a first degree in mathematics as a pre-requisite? For many countries, so stringent a requirement may indicate that in selecting the subject matter adequate attention has not been paid to the available human resources. This is not to suggest that the mathematical knowledge of the pool from which recruitment can be made is the upper limit of the content of a school mathematics curriculum. The minimum mathematical background that can be stipulated depends among other factors on the amount of time available in the teacher education programme for teaching mathematics.

27. With respect to subject matter forms questions such as the following arise:

Are teachers aware of the different forms available?

Are they able to decide which form is appropriate in a particular context?

The amount of guidance that is necessary is very likely to vary from country to country. In Ceylon it has been considered necessary to give relatively explicit suggestions in this regard. The scheme of work in mathematics referred to earlier indicate whether the teacher should state or explain some fact or principle, get pupils to discover some generalisation through a selected set of activities, etc.

While the giving of suggestions is a necessary stop, it is by no means sufficient to ensure that they are made use of. Teachers in fact may not be receptive to the suggestions. It devolves upon teacher education programmes to develop an understanding and an appreciation of subject matter forms and skill in practising them.

28. With regard to selection it will be agreed that an explicit awareness of this component is not generally expected from the novices. In fact teacher education programmes may like to assume that with respect to knowledge about this component the cognitive, affective and psychomotor potentials are zero. In practice, this may not be the case. It may well be that the potentials are negative[22]. Their sojourn as students in the education system may have, for example, led to the

development of undesirable attitudes about a particular type of subject matter form. They might be too prone to generalise on the basis of their own narrow experience.

29. Similar remarks would apply to the other components. Knowledge of the components of a lesson, except subject matter, can be regarded as being characteristic of teachers. Hence such knowledge should be developed through teacher-education programmes.

30. The deliberate choice of lesson forms can only be advantageous to the giving of the lesson.

Undoubtedly teachers practise these different forms, perhaps without an explicit awareness of them. An explicit awareness can result in their being used with greater effectiveness. Teachers should know that there are different types of lesson forms and further that during the same lesson, the lesson form could change rapidly from one type to another. Against the background of such knowledge, teachers may decide what is the best form to use for the expected situation within the resources available.

31. Media are more easily identified than other components of a lesson, but, perhaps, teachers may not be aware of the many choices possible even with limited resources. While instructions about media in general is desirable, at least in that teachers would know what are possible even if they are not available, a problem to consider is the extent to which such instruction is helpful in giving the kind of lessons that teachers are actually called upon to give.

32. Location and grouping force themselves upon the attention of the teacher when he is planning lessons about such subject matter topics as "plan-drawing" or "measuring inaccessible heights". But even if the subject matter topic is to be "solution of linear equations in one variable" and the subject matter form is to be "drill", to know that different groupings are possible may lead to a more effective utilisation of time and other resources. Perhaps insufficient attention has been paid to the contribution from this component.

33. That students should get an inkling of what is to follow, an awareness of the direction in which the teacher is leading them, has been recognized as an element of good teaching. Selection of cues, however, is not the whole of the selection of influence techniques. Knowledge that other components can be varied over a wider range than was earlier thought possible adds a new dimension to the teacher's repertoire. The other category of influence techniques including incidental evaluating techniques such as being sensitive to a perplexed look, are perhaps not selected beforehand. Skilled teachers use them as part of their routinised skill.[23]

Communication and Execution

34. At Level I the main purpose in planning a lesson is to effect a successful communication of subject matter. (If the lesson is a demonstration lesson for the benefit of would-be teachers then the attempt is not mainly to communicate subject matter but mostly to communicate the other components). All of the previous activity is judged by the adequacy of this performance. It will be accepted without

argument that explicit planning is not a sufficient condition for successful execution. Some may even assert that explicit planning is not even a necessary condition. While there might be individual teachers for whom this is apparently true, it is assumed that such is not the case for a teacher in an education system. What then are sufficient conditions for a successful performance? As far as the present writer is aware the answer to this is not completely known partly for the reason that "successful performance" is still but vaguely defined.

35. Planning as described in the previous section implies a considerable knowledge of the components of a lesson. Communication and execution imply that this knowledge be usable. Recalling, during the lesson, that the chalk board should be used is not sufficient. Even being able to assess, at that point of the lesson, whether the chalk board is the appropriate medium to use is not sufficient. The teacher should have the skill of using it (if required) raised to such a level that he is not even conscious of using it. For example, as a matter of routine his writing will be big enough for all students to see and the diagrams will highlight only the relevant features. Even this however is not sufficient if the behaviour is manifested only occasionally. Such behaviour should be characteristic of the teacher. To state this more precisely, knowledge of the chalk board should be raised to the highest levels of cognitive operation, psychomotor execution and affective involvement[24]. For other knowledge that the teacher has to achieve the specifications would indeed be different. These must be known if the training is to succeed.

36. The periods of practice-teaching which are a part of present pre-service programme are no doubt meant to develop the skill of teaching. A part of training programmes in other fields is to stage quality performances for the novices to observe, analyse and perhaps imitate. The would-be doctor, long before his period of internship sees masters at work. The masters seek to lay bare their thinking for those who may profit by it. Should not would-be teachers be given an opportunity of observing a master at work?

Control and Guide

37. Some time after the teacher has begun to put his plan into practice, he would find it necessary to take further steps to maintain the effectiveness of the communication. The subject matter form planned by the teacher may have envisaged certain responses to be made by students. For example a teacher may have expected pupils to suggest that they investigate the angle-sum property of quadrilaterals. If the expected suggestion is not forthcoming the teacher will have to either change the subject matter or change the subject matter form by stating the result and asking pupils to verify it or use some demonstration (new media) such as forming a four sided figure with triangular frameworks to encourage pupil suggestions, or take some similar step. A pre-requisite to necessary action of this sort is a sensitivity to what is happening in the classroom. The teacher must be able to obtain feedback and take whatever action necessary. A problem for teacher education programmes is to develop the skill.

Evaluation and Assessment

38. The importance of this process and its relation to decision-making and planning do not need to be emphasized. This process is not likely to occur unless the objectives of the lesson are formulated in terms of the expected student behaviour. Indeed the planning of a lesson pre-supposes such a formulation. It is of course highly probable that in teaching, the initial formulation may be modified or amended. But nevertheless the formulation of objectives in terms of expected pupil behaviour is a necessary step. This is admittedly no easy task. But if it is demanded of every teacher for every lesson that is given and moreover it is regarded as a critical process in a teacher's task then teacher-education programmes should consider whether, in the light of what is currently known, they are giving the maximum assistance to teachers in the formulation of specific objectives. Significant contributions to this aspect, in recent times, have been the taxonomies of the cognitive and affective domains evolved by Bloom, Krathwhol *et al* and the work of Mechner. [25, 26, 27]. Since then analyses of the psychomotor domain have also been published. [28, 29]. The suggestion is not that these taxonomies be adopted in toto. The suggestion is to consider whether teacher-education programmes are as concerned with the formulation of objectives as they are apparently with techniques and instruments of evaluation.

Some Significant Problems in Respect of the Practice of Teacher Education and Some Possible Lines of Approach to the Seeking of Solutions

The Teacher's task is not adequately specified

39. The fact that teacher education programmes are often classified according to subject and grade level is an admission of the belief that some information about what the teacher has to do should be available, if he is to be educated to do that. But do teacher education programmes know enough about what the teacher has to do?

Can they answer questions such as the following which relate to aspects that are not directly under their control?

What mathematics (subject matter) are teachers expected to teach?

What are they expected to achieve by teaching that subject matter?

What resources (both time and material) are likely to be available to do this teaching?

40. For the moment it is assumed that subject matter can be relatively easily identified using such materials as syllabuses, approved textbooks, evaluation instruments, etc. What is to be achieved by teaching that content is not easily found in sufficiently specific terms. For example consider the subject matter topic "Theorem of Pythagoras". Some possible outcomes from teaching the theorem of Pythagoras are as follows:

(i) Students can recall the verbal statement of the theorem.

(ii) Given a right-angled triangle they can apply the theorem to calculate the magnitude of one side given the magnitudes of the other two sides.

(iii) In addition to (ii), students can select the instances where the theorem of Pythagoras is the appropriate one to use.
(iv) In addition to (iii), students can prove the theorem.
(v) In addition to (iv), students appreciate what it is to prove a theorem.

Still other outcomes may be given or those stated above may be more finely resolved. But they are sufficient to draw attention to the fact that they require lessons of different types. For example a subject matter form (exposition, drill) and a lesson form (teacher communicates subject matter) which might be very appropriate for achieving outcome (i) above are undoubtedly not the most appropriate for achieving outcome (v).

41. The above illustration is intended to show that knowledge of subject matter, divorced from the objectives the teaching of them is expected to achieve, does not provide a sufficiently clear idea of what teachers are actually expected to do. This in turn makes teacher education difficult because what is wanted is not always perceived by the teacher education staff with sufficient clarity.

42. There are possibly many ways of specifying the task of the teacher so that not only will he get a better idea of what he has to do, but also the teacher-education programmes will have their objectives set out in a behavioural idiom. The level of specificity and the amount of detail may possibly vary from country to country. It is suggested that an adequate specification satisfy the following requirements:

(a) The subject matter should be identified.
(b) A preferred sequence of teaching the subject-matter should be indicated.
(c) Guide-lines as to preferred methods of teaching the selected subject matter (subject matter form, lesson form, media, grouping) should be provided.
(d) Outcomes of each unit of teaching should be stated.
(e) Evaluation instruments for each unit of teaching should be provided.

It is only when such specifications are available that teacher education programmes can make tentative decisions as to what knowledge of what components are required at what levels.

43. Apart from his role as leader at Level I, the teacher is also a participant at Level II of the education system. *Adapting and modifying the specifications of his task to suit the local requirements* can take place through effective participation at this level. Teacher education programmes should consider whether they have taken adequate notice of this role.

Inadequate co-operation between curriculum-design staff and teacher-education staff

44. A major part of the output of a curriculum-design staff can in fact be regarded as specifications for teaching. Ideally it would appear that the curriculum-design staff and the teacher-education staff should be one and the same. This is practically not possible since the curriculum-design staff requires personnel such as subject matter specialists, etc. Invariably the curriculum-design staff is the very much smaller group.

Failure to communicate can affect both groups adversely. A possible solution, of course, is to have some individuals who are members of both groups. This would still leave out a large part of the teacher-education staff. A better solution might be for the teacher education staff and curriculum-design staff to assume the Level I role at regular intervals.

Problem of selecting novices

45. There is a general belief that the more intelligent and more able people do not take to teaching. Assuming that this is true, for the high ability groups as currently identified, the reasons for this state of affairs might be any one or both of the following:

(i) The role of a school teacher has no attraction for the high ability group.
(ii) The salary of a school teacher has no attraction for the high ability group.

46. At first sight possible solutions seem to be to make the functions of the teacher and his salary more attractive. In examining this rather obvious solution it is relevant to note that school teachers are required in large numbers and in widely scattered locations. The problem is not the staffing of a particular class or school but the whole of a nation's schools. The solution of making the role of the teacher sufficiently attractive does not appear to be a practicable one considering the large number of teachers required. There is no doubt that every country has its quota of able men and women who have taken to teaching despite the financial loss it may have entailed. But it is the fact that the number of such people is not a significant proportion of the teacher population that has led to the problem under consideration. The other solution of making the salary attractive is not directly within the competency of teacher-education institutes. If teachers are paid a lower salary it may not be entirely a question of large numbers but a lack of genuine appreciation by society of the difficulty and importance of a teacher's task. A long term solution is for teacher-education programmes to seek to develop this appreciation.

In the above submissions it has been assumed that the high ability group as currently identified is the best source from which to recruit. A specific example of this is the belief that a person with a good honours degree in mathematics is likely to be a better mathematics teacher for a particular grade than another with perhaps no mathematics at the degree level. If this belief means that the person with the higher academic qualification has the better potential to acquire the other competencies that the skill of teaching demands, then the belief is a reasonable basis for action. If on the other hand it is believed that the one who has a good knowledge of the subject can somehow or other, without professional training, 'put it across' to students such a belief can only retard the growth of teaching as a profession. Hence even if recruitment to the teaching profession is from the high ability group an examination of teacher-education programmes is called for to ascertain how adequately they can discharge their function. It would seem therefore, that a reasonable course to pursue is to assess whether teacher-education programmes are making the best use of whatever resources are currently available to them.

Concluding Comment

48. That systematic improvement of teacher education programmes must be preceded by a clear recognition of what they are to achieve is almost a truism. This paper attempts to present the view that the objectives of a teacher-education programme be interpreted in terms of the expected behaviour of teachers, described with the appropriate degree of specificity and concreteness. An analytical framework has been proposed to study the task of the teacher. This could be used by teacher-education staff to decide what knowledge is required at what levels. Such a step is one of the essential preliminary steps to an effective programme of selection and education of teachers.

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Supplementary material provided by Mr. D.A. Perera

Suggested Mathematics Content for Pre-Service Training of Teachers

Assumed Background

1. It is assumed that those who are recruited to the pre-service training course for primary school mathematics teachers have completed at least a secondary school course in what is generally regarded as traditional mathematics. Recruits for the secondary school teachers' course, it is assumed, have completed at least an A-Level course of a similar type. Most secondary school mathematics courses would have been geared to the development of computational skills such as the following:

(i) fundamental operations on rational numbers;

(ii) operations involving percentages, averages, ratio, proportion;

(iii) operations on numbers where the numbers are used (along with the appropriate units) to represent quantities such as time, money, length, area, etc;

(iv) transforming given arithmetic and algebraic expressions with such techniques as simplifying, factoring;

(v) solving linear equations in one or two variables, second degree equations in one variable, graphical solution of equations;

(vi) expressing given information using mathematical symbolism, solving verbal problems;

(vii) using mathematical instruments, constructions using straight-edge and pair of compasses;

(viii) proving selected theorems from Euclid and doing original exercises based on them;

(ix) using sine, cosine and tangent to solve right-angled triangles, skill in using mathematical tables.

2. These skills would have been developed in terms of a large body of content such as properties of plane geometric figures, tables of weights and measures, formulae for area, volume, perimeter, etc. An understanding of the structure of the subject may not have received due attention. In addition to the above, an A-Level course would have dealt with co-ordinate geometry of two dimensions, integral and differential calculus and more advanced algebra and trigonometry. Synthetic solid geometry of three dimensions would also have been done.

Some Assumed Objectives of a Subject-matter Course at a Training Institute

3. (i) Awareness and appreciation of the development of mathematics and mathematical modes of thought as a part of the development of human culture.

(ii) Understanding of fundamental mathematical concepts such as sets, relations, number, operations, numeration system, variable, proof, etc.

(iii) Awareness of the abstract nature of mathematical concepts. Understanding of a mathematical model.

(iv) Acquisition of a unified view of some of the different areas of mathematics.

Suggested Content

4. A difficulty in specifying content in the form of subject matter topics is that the latter are open to being interpreted over a very wide spectrum. Another difficulty is that even if the content were to be made fairly specific, the same content may be taught at different levels of cognitive operation. However a formulation of the content in terms of subject-matter topics could be a step in the evolving of more adequate guide-lines.

a. For primary school mathematics teachers

Idea of a set. Subset, disjoint sets, null set. Venn Diagrams. Correspondences. Equal and Equivalent sets. Cardinality of a set. Set operations.

Counting numbers, integers, rational numbers. Operations on numbers. Ordered pairs. Mappings on rational numbers. Operations as mappings. Idea of the development of the number system. Awareness of irrational and complex numbers. Modular arithmetic.

Variable, domain. Relation and function. Range of a relation. Argument and value. Equivalence relations. Inverse relation. Open sentences. Propositions. Equations and inequations. Conditional equations, inequations. Solution sets. Identities. Graphing as a one-to-one mapping. Graphing on the number line and co-ordinate plane. Graphs of solutions, set of equations and inequations.

Geometrical figures as sets of points and lines. Elements of Euclidean geometry including the theorem of Pythagoras. Intuitive study of transformations of geometric figures. Invariants under transformations. Symmetry and similarity. Elementary trigonometric functions. Elementary ideas of probability and statistics.

b. For Secondary School Mathematics Teachers

In addition to the content listed in 4a, the following:

Algebra of sets. Boolean algebra. Equivalence and order relations. Algebraic operations and their inverse. Semi-group, group, ring and field. Understanding of algebraic structure.

Set of positive integers. Principle of mathematical induction. Integers, rational numbers, irrational numbers, real numbers, complex numbers.

Simple series, finite and infinite. e^x, e^{ix}, $\log x$, $\sin x$, $\cos x$.

Solution of a system of linear equations. Matrices, determinants. Approximate solutions of equations of higher degrees.

Historical development of Euclidean geometry. Geometric figures as models for physical objects. Elements of a deductive system. Awareness of other geometric systems. Use of matrices in studying transformations.

Euclidean vector. Elementary idea of a vector space.

Upper and lower bounds. Infinite limits and finite limits as $x \to +00, -00, a - o, a + o, a$.

Neighbourhoods. Simple theorems on limits.

Differentiation. Differentiability at a point. Differentiability in an interval. Differential. Theorems on differentiation. Maxima and Minima as local properties. Integration as the inverse of differentiation. Areas. Extension of the concept of integration.

Elements of a deductive system. Categorical and hypothetical propositions. Converse, inverse and contrapositive, etc. Logical connectives. Truth tables. Syllogisms. Valid Inference.

c. For teachers of other subjects at the secondary school level

(It is assumed that these teachers have completed at least a secondary school course in mathematics as described earlier. With respect to objectives it is suggested that the main emphasis be on 3(iii) and 3(ii)).

Idea of positive integers in terms of equivalent sets. Development of the number system. Numeration system with non-decimal base.

Variable, domain. Relation and function. Range of a relation. Equivalence relations. Open sentences. Propositions. Conditional sentences. Identities. Awareness of an algebraic system. Elements of a deductive system. Categorical and hypothetical propositions. Converse, inverse and contrapositive etc. Logical connectives. Truth Tables. Syllogism. Valid Inference.

REPORT OF WORKING GROUP B.2

Chairman: Professor C.O. Taiwo (Nigeria)

Supply

49. The members of the group first discussed current practices in their respective countries with regard to the supply and training of teachers of mathematics. It was recognised that there is a shortage of competent teachers of this subject at all levels. In some countries, the shortage is of qualified teachers; in all there is a dearth of good teachers, competent in action. In several developing countries, there are many teachers who have not been exposed to the new ideas and yet have to teach mathematics. In particular, there are primary school teachers who have little or no knowledge of the subject matter or of the approach which is now recommended in the teaching of mathematics in the primary school. There are tutors in the training colleges who themselves need to become acquainted with the new ideas and approach.

50. Various reasons were suggested for the shortage of teachers of mathematics. Throughout the world there is an increasing demand for mathematicians in administration, industry and commerce, where conditions of service are more attractive. The status and salary of the school teacher are less favourable than that accorded to comparable positions open to mathematicians outside teaching. The role of a school teacher appears to have little attraction for the high ability group, some of whose members might make good teachers of mathematics.

In developing countries some teachers receive no initial training at all. This is because of the expense to the countries not only of the training itself but also of the higher salary scale on which trained teachers must be paid. There is also some 'brain-drain' arising from students who find their way to developed countries, achieve higher qualifications, and remain in those countries where conditions of employment are more favourable. Loss also occurs when qualified teachers of mathematics now in service in their own countries seek more lucrative forms of employment.

It was recognised that there is a core of able and competent men and women in every country who have remained in the profession despite financial loss. It is clear that efforts should be made to examine further the reasons for the shortage of competent teachers of mathematics, and that practical steps should be taken to ensure a good supply of such teachers and to retain them in the profession.

51. The tendency of some of the abler students in mathematics to go into professions other than teaching was discussed. Careful consideration should be given to the training of those students who have so far not demonstrated the highest mathematical ability, provided they have the qualities required of prospective teachers.

Selection procedures

52. Entry qualifications into the training colleges for primary school teachers vary. In some countries a secondary school certificate (usually following eleven years of successful schooling) is required; in others slightly lower qualifications are required and the length of professional training is prolonged to three or four years. In other countries, the Primary School Leaving Certificate (obtained after six to eight years of schooling) is the required minimum qualification. The Group considered that a desirable entry qualification at which to aim for all teachers is the Secondary School Certificate or passes in several subjects at the General Certificate of Education ("O" Level). A pass in mathematics, while desirable, is not essential. In countries where the average student applying to enter the Training College does not possess the desirable minimum entry qualification, it is all the more important to have tutors competent to handle mathematics in the manner recommended in the Conference Report.

53. Methods of selection in the different countries were discussed. It was agreed that some form of interview in addition to an examination was desirable in selection for training. It was suggested that the interview procedure might consist of two parts:

(i) a personal interview; and
(ii) a group interview, which would give an opportunity to see how candidates react individually and in groups.

54. It was recognised that some countries found it convenient to allow Principals of schools to administer a standardised interview to their pupils who wished to enter a Teaching Training College; the comments are then passed to the respective Teacher Training College for final selection. But it was strongly urged that the Training

Colleges should be well represented in any selection procedure for students, and that the final selection of candidates should be in the hands of the Training College.

55. It was agreed that candidates who are older and have had some responsible experience since leaving school, were likely to prove good candidates for a course of training.

56. It was agreed that professional training for university graduates is highly desirable. Among other things, it assisted them to form good attitudes to their profession, to learn about children and their ways of learning and to communicate meaningfully and effectively with their pupils. It should lead to a deeper understanding both of the aims of teaching and of the problems usually met in the classroom.

Initial training

57. Training in colleges for primary school teachers usually incorporates professional training with academic study of a number of subjects including mathematics. There should be a mathematics course for all students which would give them a new insight into the mathematics that they would be expected to teach and the methods they might use, and a familiarity with experiences through which children learn. There should also be an optional specialist course designed to stimulate students in their own mathematical pursuits.

58. At least some training colleges should provide courses for students without a university degree who intend to teach mathematics at a secondary school. Such a course could be a straight three-year course (as in some countries), which might lead on, immediately or at a later stage, to a Degree Course; or it could be a two-year course followed by a Mathematics Specialist Course of one year. Other variations are possible. Courses might be organised in which students specialise in two main subjects, one of which might be mathematics and the other a language (English, French, a local language, etc.) or some other Arts or Science subject.

59. Two schemes were discussed for training university graduates to teach mathematics. The first was a degree followed by professional training; the other was a degree incorporating professional training such as the B. Ed., B.A., (Education) or B. Sc. (Education), where the precise qualification depends on the university regulations applicable to the candidate. It was considered that each scheme had its own advantages. Both schemes are already being offered in a number of Universities and Institutes of Education. Some degree courses might be structured to cover mathematics and some Arts subject in order to offset the shortage caused when science graduates with mathematics move into other forms of employment.

60. Consultation between mathematics tutors in different colleges and countries is highly desirable. It is therefore suggested that opportunities be given to those teaching mathematics at colleges to benefit from bursaries or study leave periodically, in order to study new methods and new topics in teaching mathematics.

61. People with experience of experimental teaching could be invited to discuss their methods with the tutors of Training Colleges. Such discussions could be accompanied by demonstrations using local pupils, and materials available in the environment.

62. In planning new college buildings, provision should be made for special mathematics rooms, mathematical laboratories and appropriate storage spaces. In the same way, furniture such as tables and chairs, flat topped desks, should be designed to permit practical work and give mobility.

Subsequent training

63. Teachers with little knowledge of mathematics could be assisted to learn more of the subject and newer teaching methods through subsequent training courses spread over a reasonable period (one or two years). There are various ways in which teachers could receive the help they need:

(i) At centres where they could meet to receive guidance, exchange views and experience and discuss the problems which arise in their classrooms.

(ii) By working with one another, observing how the more experienced teachers handle their class groups.

(iii) By meeting to make simple apparatus and to experiment with local materials.

(iv) By reading books in order to keep up-to-date with recent developments in mathematics teaching. For this purpose the establishment of collections of mathematical books in school and class libraries is essential. Such reading is particularly important for secondary school teachers.

(v) Publications, especially teachers' journals or bulletins, should be used more widely as a means of informing teachers of any new ideas in approach to and content of mathematics.

64. Secondary School Teachers of mathematics have special needs which could be met in various ways:

(i) They should continue reading mathematics as well as reading about recent developments in teaching.

(ii) Facilities for further training courses which some teachers require should be made widely known. Examples of examinations available are the Mathematical Association's Diploma in Mathematics organised by the Mathematical Association of Britain, and the External Degree of the University of London.

(iii) Universities and Teachers Colleges should be encouraged to provide suitable evening courses or other part-time courses in mathematics. Such courses should aim at assisting teachers to acquire a greater knowledge of the subject and to enable them to teach mathematics more effectively and meaningfully. Where feasible, university courses biassed towards Education should be offered to teachers. Teachers could apply for entry to these courses under the Mature Student Admission Regulations which are a feature of most universities. Such regulations should give special consideration to teaching experience.

(iv) University graduates in Arts subjects who find themselves having to teach mathematics should be assisted by means of suitable courses to acquire a knowledge of both the subject matter and the newer approach to the teaching of mathematics.

(v) Journals specialising in mathematics could be of great assistance to secondary teachers, and it is urged that efforts should be made by teachers to have such journals established in countries where they do not exist.

(vi) Where practicable, efforts should be made to make publications from other countries available to teachers in order to stimulate and introduce new ideas.

(vii) Mathematical Associations should encourage imaginative teachers to get together, discuss ideas, and influence developments as was suggested in a paper submitted by Britain.

(viii) Exchange programmes by which teachers from other countries work with local staff and students should be encouraged. Care should be taken to ensure that exchange teachers become familiar with the environment of the pupils and can make use of local materials.

Conferences and workshops in mathematics

65. Workshops in mathematics provide opportunities for teachers of various age groups to meet together to learn mathematics in an active way themselves, to plan work for their classrooms, and subsequently to meet and discuss the pupils' work resulting.

Central planning is needed to organise workshops for tutors and leader-teachers in the first instance. These people would then act as leaders in workshops for teachers. Success of such workshops depended on the co-operation of the supervisors, principals and inspectors of schools who should at all times be made aware of the plan in order to enlist their support. "Follow-up" after the completion of courses is very important and should be incorporated into any general plan. Sometimes authorities needed to provide incentives, such as payment of cost of travelling in order to generate initial interest in such workshops. Where necessary, experts could be invited from developed countries for periods of six months to one year to assist in conducting such workshops.

Another form of subsequent training is the single School Training Session which concerns only the teachers in one school. This is especially useful where group centres are not feasible on account of long distances involved or transport difficulties. The day commences with a preliminary discussion by the entire staff of the school's mathematics course in outline and of the underlying philosophy of the course as a whole. Attention is then focused on a series of different classes. Each time the class teacher is in action, teachers of the classes immediately above and below, as well as the principal and any expert available, are present.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP B.2

Survey and Comments

66. Discussion centred on the differences in the opportunities open to primary and secondary teachers, in regard both to training and to salaries. In several countries expense made it virtually impossible to give initial training to all primary teachers because in addition to the cost of the training the countries could not afford the higher salaries payable to trained teachers. Usually salary depends on qualifications and not on the type of school in which the teacher is engaged. The more highly qualified teachers are to be found in secondary schools; the untrained are appointed to deal with the large numbers entering primary schools. Delegations stressed the importance of investing in primary education and employing trained teachers if the educational level of a country is to be raised, because at that stage teachers are laying the foundations of all future learning whether academic or practical.

67. Methods of selection vary considerably. Delegates were anxious to ensure that, at entry to training, primary teachers had an adequate general education and secondary teachers of mathematics had good mathematical knowledge. In spite of the difficulties in listing the qualities required in a teacher, delegates believed that the final decision on a candidate's suitability should be made by the college staff.

68. Delegates emphasised the importance of mathematics courses during the initial training of primary teachers. The basic ideas of modern mathematics should be included but it was also essential that students should themselves experience learning mathematics through exploration and experiment. Attention was drawn to the possibilities of apprenticeship schemes in which students alternate between paid employment in a school and study terms in a Department of Education.

69. Many different ideas for the necessary subsequent mathematics courses were mentioned. It was recognised that all resources would have to be utilised to provide the workshops, mathematics centres, handbooks, etc. needed to prepare teachers to introduce new methods and content.

CHAPTER VIII

Resources for Learning Mathematics

(Including textbooks, films, radio, television, programmed learning, etc.)

Lead Paper by Professor A.L. Blakers, M.A., Ph.D., B.Sc.,
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Introduction

1. I am both pleased and mystified by the invitation of the Organising Committee to address this Conference on the topic "Resources for Learning Mathematics". I am pleased, because the invitation has given me the stimulus to gather together and think about some of the many books and papers on this topic, and because of the unexpected opportunity to meet and exchange ideas with so many Commonwealth leaders in mathematical education; I am mystified, because I do not believe that I have ever made any significant contribution to the development, use, or assessment of any of the resources mentioned in the title of my topic, and I cannot imagine why I should have been asked to speak on this topic to a conference of experts.

2. It is inevitable that some of what I say will overlap with other prepared papers. This is so because the training of teachers, and evaluation, are the topics of other plenary sessions. But teachers are among the most important of our resources, and the usefulness of a teacher as a resource is obviously related to his training; moreover the effectiveness of any resource must be determined by evaluation, and it is difficult to divorce the evaluation of programs from the evaluation of resources.

3. It is also inevitable that, due to the need to prepare this paper in a few brief months, there will be significant omissions of important material. This, however, is no cause for concern, and can be considered as a stimulus to the appropriate Working Party to put things right by filling in the gaps.

Terms of Reference

4. When approached to prepare this paper on resources, my "brief" consisted solely of the assigned title, and a suggestion that I give relatively less attention to the more sophisticated and expensive new technological developments. I have interpreted my brief very liberally, taking advantage of the "etc." which came with the title, and also of the fact that the title uses the expression "*learning* mathematics", rather than the expression "*teaching* mathematics". Thus in addition to the specific resources listed in the title, I intend to include our environmental resources (physical, biological, socio-economic, technological, cultural, etc.); books other than textbooks; computers; and that most important resource, the teacher.

This extended list is by no means complete (we haven't even listed "money"!); nor, of course, are the items in it independent.

Educational Media

5. Many of the resources I have mentioned above (possibly all of them) are included in the currently popular term "media". Perhaps I should have said "educational media", but the adjective is probably redundant: surely all media are capable of being used for educational purposes. In this connection I would like to refer you to the recent book *Understanding Media*[1] by Marshall McLuhan. The title notwithstanding, I do not know anyone who claims to understand a great deal of what McLuhan is trying to say, and yet I have the feeling that it is important, and that it will stimulate further thought and work which will be highly relevant to the topic of this paper.

Resources and Learning Theory

6. In comparatively recent times psychologists have made great strides in their studies of human personality; anthropologists and psychologists have made us very aware of the relationship between personal development and environment; scientists and technologists have steadily increased our ability to modify our environment; and educational researchers have been actively exploring the relationship of environment and personality to the learning process. A variety of learning theories have been formulated, and serious attempts have been made to evaluate them. Some of these theories (e.g. the theories of B.F. Skinner: see below), have had a considerable influence on the development and use of some of the most "modern" learning resources: in particular, programmed instruction. No doubt this situation will continue, with our understanding of the learning process, and a more sophisticated and successful use of resources, developing hand in hand.

Resources for learning Mathematics

7. We have not come to this Conference in order to speculate about the uncertain future, but rather to face up to the realities of the present. We wish to exchange information, experiences and opinions on those resources for learning mathematics which already exist, or are likely to exist in the near future. In introducing these discussions, it therefore falls to me to say something about a number of specific resources. In discussing these separately, I will attempt to say a little about their current state of development and use, and to include opinions as well as facts, in the expectation of provoking some of you to disagree with me. The list is not intended to be categorical, nor are the items in it independent of one another.

Our Natural Environment

8. All of the specific resources which we shall consider below form part of our total environment. This environment is by no means the same for all of us – in fact every one of us is a genetically unique individual, and each of us is subject from birth to a personal and unique set of environmental factors. Thus at every stage of our development our potentiality for learning (in mathematics as in all else) is unique, and, ideally, this should be exploited by a unique use of available resources. However all existing systems of education operate by considering categories of individuals, grouping them by the use of such criteria as geographical location, age,

sex, apparent success in previous education, and so on. It is convenient to consider a rough division of our total environment into that part which is natural, and that part which is man-made. The "mathematical content" of our natural environment is much the same for all of us, wherever we happen to be located on this little planet. The shape of the sun; the apparently changing shape of the moon; daily and seasonal patterns of shadows; regularity and symmetry in crystals, leaves, flowers, and seeds; experiences with water, such as reflections, waves and ripples; – these are freely available to all. We know that many of these things have had a significant effect on the historical development of mathematics, and it is not unreasonable to believe that we should be able to exploit them as resources for the learning of mathematics.

9. Although it is probably not relevant to the topic of this paper, we should remind ourselves that there is a deeper relationship between mathematics and the natural environment, and that until very recently this environment has been virtually unchanged during the evolution of man. There is little doubt that among those genetic variations of prehistoric man (and his ancestors) which had survival value in relation to the environment, there must have been variations related to abilities which we would consider to be essentially mathematical. In other words, the fact that man has evolved as an animal with considerable mathematical capability probably reflects the value which this type of ability has had for survival in a competitive environment.

Our Man-Made Environment

10. Under this heading we include all aspects of our total environment which are, directly or indirectly, the creation of man. This includes not only those specific products of our manufacturing technology which are mentioned in the title of this paper, but also such other man-made creations as our homes; our cities; our social, political and economic systems (including our use of money); our many devices for the transportation of goods, information and ideas; and, of course, the many useful and useless gadgets which flow from the application of our advancing science and technology. The natural environment has played a well-known and significant role in the historical development of mathematics, but the man-made aspects of our environment (and our need to understand and further develop them) have now assumed a far greater significance in relation to current mathematical developments, and to current curriculum developments. While we seem to be moving slowly in the direction of greater uniformity in the technological aspects of our societies, there are still very great differences from one place to another. Whether or not this implies the need for differences in mathematics curriculum will, no doubt, be discussed in other sessions; but there seems little doubt that these environmental differences deserve serious consideration in relation to the use of resources for implementing whatever curriculum is used.

11. Let me give a simple example. In my country, it has been pointed out to me that the mathematical progress of children in rural areas is indistinguishable from that of city children through the primary grades, where the mathematics studied is almost wholly arithmetic; but that apparently significant differences begin to appear at the secondary level, with urban children achieving greater success than rural children. It is conjectured that the environment of urban children (includ-

ing, of course, the attitude of parents, friends and especially, the peer group) makes the acquisition of mathematical knowledge more desirable, and apparently more obviously relevant to the needs of the community. The same curriculum is almost certainly needed for all of these children, in view of the uncertainty concerning their future careers. But one naturally wonders whether or not the different environmental influences could be compensated for by the use of different teaching strategies.

12. The profound influence of our cultural environment on our personal development is, of course, well known to psychologists and to anthropologists but we sometimes tend to overlook it in relation to the learning of such a universal and apparently culture-free subject as mathematics. It is my personal belief that not enough attention has been paid to cultural differences, both in the design of curricula, and in the use of various resources for learning mathematics. It might be hopelessly visionary to contemplate a time when the design of mathematics curricula and the use of learning resources will be fully adapted to the needs and potentialities of each individual, but it is not quite so unrealistic to suggest that more attention (especially in the use of resources) should be paid to some of the more obvious cultural differences. This implies, for example, that books designed for use in one society should be "culturally translated", as well as verbally translated, in order to adapt them for the needs of another – an innovation which has already been found worthwhile for culturally distinguishable groups within one country, as well as between countries with obvious cultural differences. (See for example[2], and many other American reports on programs for culturally disadvantaged children). Of course, many other dichotomies are possible in addition to such well known ones as rural-urban, affluent-poor, and culturally "normal"-culturally deprived. One which has received a great deal of attention in recent years is related to the relative state of technological advancement, classifying countries rather roughly as "developed" and "developing". I recently read a charming book[3] about the experiences of two teachers from one of the most technologically developed countries (the United States of America) in teaching elementary mathematics to children from a particular tribal group in Liberia. This delightful and thought-provoking study has implications which go far beyond the immediate experiences of the authors: it suggests most strongly that the teacher should understand the culture in which he is working, and how this affects not only the suitability of what is taught (the curriculum) but also the choice of a teaching strategy (including the use of learning resources) which is likely to prove most effective.

13. This attention to cultural differences is important in time, as well as in place: the cultural environment of any part of the world varies in time, and in most areas the position today is very different to what it was fifty (or even twenty) years ago. Dr. Robert Davis has drawn attention to the tremendous significance of these time-linked cultural changes, in his fascinating and provocative booklet[4], *Mathematics: The Changing Curriculum.* This booklet is essential reading for anyone who is interested not only in seeing where we are (in mathematical education), but who is also interested in seeing some of the directions in which we are heading. Dr. Davis points out that the technology of the 1960's and the use of mathematics in it, are vastly different to what they were in the 1940's, and he suggests that all of

our recent "reform" efforts, even if successful, might only result in making our curricula relevant to the needs of the 1940's. He suggests that the technological development of (American) society requires a rate of change of mathematics curriculum and learning procedures which is considerably greater than the present rate.

14. I shall not carry further this discussion of environmental factors, but I hope that what I have said will make clear my belief that our man-made environment has had, and will continue to have, a great influence on the developing nature of our mathematics, on the choice of curriculum, and on the choice of strategy for teaching (learning) this curriculum, and my strong conviction that both the choice of curriculum and the use of available learning resources should take into account the nature, and the changing nature, of the cultural background of our students, and of the future environment in which they will live. If we can discover how to do this successfully, then it will be possible to exploit the environment itself much more than we do as a resource for learning mathematics.

15. Of course, it is rather easy to assert what should be done. We might all agree on the importance of cultural factors in curriculum design and in learning strategy, but find great difficulty in turning this agreement into action. However, the report[2] referred to earlier, indicates that some progress is being made in one country (the United States) in the understanding and amelioration of the cultural deprivation which results from relative poverty. I recently read in the Annual Report (1967) of the Carnegie Corporation of New York, of a quite different project (a joint project of Syracuse University and Makerere College) designed to discover whether or not current psychological theories of child development and learning (which have been predominantly based on studies of children in Western societies) are, in fact, applicable to non-Western cultures. Studies such as these indicate that some progress is possible, and they hold much promise for the future.

Specific Resources: The Teacher

16. During the past fifteen or twenty years there has been an almost unprecedented amount of activity aimed at the "improvement" of school mathematics. Initially much of this activity was directed towards curriculum reform, with such landmarks as the creation of the Commission on Mathematics in 1955, and the publication of its report[5] in 1959; the establishment of the School Mathematics Study Group, in 1958; the Royaumont Seminar (1959) of the Organisation for European Economic Co-operation, and its controversial report[6] *New Thinking in School Mathematics;* the call to arms by Professor Bryan Thwaites (1961) in his Inaugural Address[7], leading to the Southampton Mathematics Conference[8], and to the School Mathematics Project; the imaginative work of the Cambridge Conference on School Mathematics (1963) and its stimulating report[9] *Goals for School Mathematics;* and, more recently (1966) the Ditchley Conference[10], a joint Conference of the School Mathematics Project and the Cambridge Conference.

17. This listing is, of course, very incomplete. Reform movements exist in virtually every country, and there can scarcely be a school anywhere whose mathematics program has not been affected by the activity of the last decade. (A fuller

picture is given by the recent UNESCO publication[11] *New Trends in Mathematics Teaching).* In most of the above mentioned reports there is relatively little mention of the role of teachers and other resources. However it was early realised by many of the leaders of the reform movements that the best-conceived curriculum would founder if due attention were not paid to the crucial role of the teacher and to the improvement of teachers themselves, as well as the improvement of other resources such as textbooks and teachers' guides.

18. The events of the last ten years have fully justified the initial concern (which is evident in many of the publications referred to above) that a major barrier to the successful implementation of any new curriculum would be the mathematical and pedagogical competence of the teacher. I am sure that many of us who are here at this Conference will have independently reached the conclusion that it is the teacher on whom we must concentrate much of our attention, if genuine progress is to be made. This is not really surprising. For it is the teacher (who is often referred to as "a filter, through which mathematical ideas percolate to the student"; or as "a mirror – sometimes cloudy! – in which the student perceives mathematical concepts") who is still the main channel for the communication of mathematical ideas to the student, no matter what curriculum is being studied. This statement is as true for classes using the many fine new textbooks that have recently appeared, as it is for classes using older books or having no textbooks at all. And, as far as we can tell at present, the role of the teacher is likely to be just as significant (even if different in detail) in conjunction with such "automated" teaching methods as programmed learning and computer assisted learning.

19. It is my own conviction that the teacher is, in many ways, the critical classroom resource, and that the teacher sets a limit to the effective use of every other resource – textbooks, films and filmstrips, programmed materials, attribute blocks, calculators, overhead projectors, number rods and other analogy devices, mathematical games and puzzles, and the many other "teaching aids" which are appearing in our classrooms. For this reason I look forward with interest to the workshop discussions on the vital topic of teacher selection and training. Unfortunately it is extremely easy to convince ourselves of the crucial role of the teacher, but it is vastly more difficult to know just what to do about it. We know that the successful teacher must have an adequate knowledge and understanding of the relevant mathematics, and as much skill and understanding of the processes of learning and of pedagogy as we can currently provide: these are the traditional components of teacher training. But there is much more to it than this. Most of us will recall personal learning experiences in mathematics (and in other subjects) where we became excited about a particular topic, and this excitement acted as a catalyst in the learning of the topic, to the extent that we were not conscious at all of any "effort" to learn. We would like to be able to excite all of our teachers about their subject and its teaching, and to have them communicate this excitement to their students. Perhaps one day we shall discover how to do this.

Books

20. Let me confess at the outset what will probably become apparent as I proceed: that I am a book lover. Books have been with us for a very long time,

and their role as carriers of various aspects of our human culture is well known. (Consider, for example, the historical significance of such books as the Bible, the Koran, and Euclid's Elements). The two great storehouses for the accumulation and transmission of mathematical knowledge have been people and books.

21. In considering the significance of books as a resource for learning mathematics, it is necessary to remind ourselves of the dramatic change in the status of books brought about by the development of modern, movable-type printing in Western Europe in the fifteenth century. This technological breakthrough made it possible to produce many identical copies of the same book at a great reduction in cost as compared with earlier methods. It also made possible a rapid development of mass education. The reasons why such a development did not occur for several hundred years are complex, and beyond the scope of this discussion; but it is safe to say that, had printing not been invented, the development of mass education in the last hundred years would have been very different from what it has been. To realise this we have only to reflect on the extent to which our personal development – in mathematics as in other areas – has been assisted by our use of books: textbooks, reference books, journals, and so on.

22. Books and printing have been with us for so long that it might be thought that, except for changes in content, nothing of current significance could be said about the role of books. But I do not believe that this is the case: authors and book publishers have not stood still. One interesting development of the last decade (closely associated with curriculum reform) has been the growing tendency for the use of fairly large groups of writers, rather than the more traditional authorship by one, or by a small number of writers. This group writing is seen in the work of such bodies as the School Mathematics Study Group, the School Mathematics Project, the African (Entebbe) Mathematics Project, and, of course, many others. There is no doubt that such group writing can result in a very critical selection and appraisal of material, although it sometimes runs into problems of style. I am sure that the potentialities of group writing have not been fully realised in most of the recent projects, due to rather severe restrictions of time and money. But, given an adequate supply of both, it seems safe to conjecture that many of the best textbooks of the future will be produced by the co-operative efforts of mathematicians, teachers, learning theorists, and publishers; assisted, of course, by feedback from the students on whom the texts are evaluated. In this connection see the remarks of Robert Davis on "Experimenting with Textbook Style"[4].

23. There have also been significant changes in recent years in the art and technique of book production. For example, consider the attractive use of art work and colour in the books[12] of the Nuffield Mathematics Project: these books are a far cry from the generally drab textbooks of a generation ago. Another innovation has been in the use of transparent overlays; for an example of this see[13]. It is not clear just how much effect a more attractive production can have on the learning of mathematics, quite apart from the aesthetic pleasure which it may give us, but I am inclined to think that it could be significant for some students; it would be interesting to see some well planned educational research on this question.

24. While talking textbooks, there is one matter which simply must be discussed: that is, how significant is any textbook in the learning process for the individual student, other than as a course-guide for the teacher? In my country, until recently, mathematics textbooks for the primary school (and, to a lesser extent, the secondary school) were little else than collections of problems. The "textbook" (if any) was not generally designed to present the subject in an orderly way which could (in theory at least) be read by the student as a supplement to his class instruction. There are signs that this situation is changing significantly, and I will be most interested to see whether these students (who are now being encouraged to read mathematics books at both primary and secondary levels) will impress us, who later teach them at the tertiary level, as being significantly more successful than our present students in their ability to learn mathematics, by themselves, from books and journals.

25. At this stage I hope that you will forgive me if I inject a somewhat more personal note, and tell you of a project on which I have been engaged for the past seven or eight years. About ten years ago, as a result of my increasing involvement with mathematics education in schools, I became aware that most schools in the region where I live (Perth, Australia) had virtually no worthwhile mathematics holdings in their libraries. (A typical situation was for a school to have a few musty and out-dated textbooks, frequently bequeathed by some long forgotten teacher). I also became aware that very few teachers of mathematics had any significant number of mathematics books in their personal libraries. In view of the rapidly growing number of suitable books (for both teachers and schools) which I knew to exist, it seemed to me that I should be able to do something to improve this situation. My first approach was to gather together a list of a little over a hundred suitable books, and I circulated this list to teachers (as members of our Mathematical Association) and to school librarians. After a year or so I made some attempt to assess what effect this had had. Unfortunately it appeared that any improvement was very small indeed. After further discussions with teachers, I suddenly realised that neither teachers nor librarians were likely to buy any significant number of mathematics books which they had no opportunity to inspect in advance. And Perth bookshops (including our University Bookshop) rarely stock very many mathematics books which are suitable for the libraries of schools and of teachers.

26. Having thus clarified my problem, the solution was relatively simple. I set about collecting a "standing display" of such books, housed in a location where they could be readily inspected (including limited borrowing) by any interested teacher. In addition, and with the co-operation of the State Department of Education, these books are regularly circulated (for a limited inspection period) in boxes of 20-30 to any interested school. (You might be interested to know that the State of Western Australia occupies almost a million square miles, and that some schools are over 1,000 miles from Perth. Thus not every teacher can drop in at the University – where the books are normally housed – after school!) Finally, a list of recent acquisitions is sent regularly to all members of the Association.

27. This collection has now grown to about 700 books. In case you are interested in starting such a project for yourselves (and I strongly recommend that such "school mathematics collections" should be established in every major centre) I have arranged

to supply each of you with a fairly complete list of our current holdings. No doubt some of this material will be familiar to you, but I am sure that you will be as surprised as I was just how much is available, and how rapidly this is increasing.

28. Perhaps I should say something about the selection of material for this collection. I began originally with the idea of sending out a brief "review" of each book, with each list of acquisitions. But almost from the beginning I had requests from teachers to put the books in some sort of order of preference; and this worried me, because I was not convinced of the infallibility of my judgment (or of anyone else's), and I did not like the thought that every school might end up with the same little subset of my collection. My concern about possibly playing the role of "censor" soon became irrelevant, because the books came in much faster than I could possibly read and comment on them, and the only books which I have excluded have been quite routine standard texts.

29. As a result of all this, teachers and schools have made their own selections, and these have shown a gratifying diversity. Some of them have certainly bought books which I would never have recommended. But this doesn't worry me, especially as teachers have told me of particular books which have aroused the interest of some of their students, and I have realised that I would have deleted these books in any censoring process. I am more than satisfied if a book which I consider dull, inelegant, and even inaccurate, can arouse the interest of any student in mathematics: interest acts as a powerful catalyst in the learning process.

30. So that some of you can see some of these books for yourselves, I have arranged to have as many of them as possible brought together in Trinidad.* I hope that some of you might be influenced to work towards the establishment of such collections at suitable centres in your own countries, and, if so, my experience might provide you with a starting point. Of course if such a collection is started, it should be kept up-to-date by the regular addition of new material. Among the sources which I find useful in helping to locate suitable books are the reviews, advertisements, and bibliographies in such journals as *The Mathematics Teacher, The Arithmetic Teacher, The Scientific American, School Science and Mathematics, Mathematics Magazine, The Mathematical Gazette, Mathematics Teaching,* and *Teaching Arithmetic.* In addition annotated lists have been published by the Association of Teachers of Mathematics (U.K.), and by the National Council of Teachers of Mathematics (U.S.A.).

Programmed Learning

31. It is fairly safe to say that programmed instruction has been one of the most controversial educational innovations of the past decade. The recent wave of interest and activity in this technique has its origin in the work of S.L. Pressey, who pointed out in two articles published in 1926 (*A Simple Apparatus Which Gives Tests and Scores – And Teaches* and *A Machine For Automatic Teaching of Drill Material;* reprinted in[14], pp. 35-41 and pp. 42-46) that a multiple-choice testing machine also performed a teaching function. However, the roots of the idea are as old as education itself, and are certainly present in the didactic methods of Socrates. In fact,

*These titles are included in the bibliography.

as Kenneth May has pointed out in his masterly studies *Programmed Learning and Mathematical Education*[15], and *Programming and Automation*[16], programmed instruction in the modern specific sense can be considered as a particular instance of educational programming, which he defines as the scheduling and control of student behaviour in the learning process. Thus all educational procedures involve programming (including self-programming) to a greater or lesser extent.

32. Pressey's work does not appear to have made any significant impact at the time of its publication, but this cannot be said of the later work of B.F. Skinner. Following the appearance of Skinner's papers *The Science of Learning and The Art of Teaching* and *Teaching Machines* (which appeared in 1954 and 1958 respectively; these are reprinted in[14], pp. 99-113, and pp. 137-172) there has appeared an enormous amount of programmed material in a variety of modes, and on a very wide range of topics.

33. From the beginning mathematics has been considered (by the promoters of programmed learning; not necessarily by mathematicians) as a suitable subject for programming, and programs on mathematical topics soon began to appear on the scene. I think that it is fair to say that mathematicians have largely been sceptical of the many optimistic claims which have been made for this new technique, and that May's two papers referred to above give a fair impression of the views of the mathematical fraternity. The reasons for these doubts are many: from the beginning the technique was oversold, both by educational enthusiasts and by commercial promoters; it was not at all obvious that work done on animals would have direct application to the learning of human beings; the style of production makes it very difficult for a mathematician to get a quick feeling for a program, as he can for a book, by skimming the content and sampling a few passages; and when he took the trouble – an extremely boring procedure – to examine carefully a programmed mathematics text by following it through in detail, he was often appalled by the mathematics he found. (This is not very surprising, since most of the early mathematics programs were written by people with no established standing as mathematicians or as mathematics teachers – people who would have been most unlikely to write a normal mathematics text). In any event, mathematicians have a healthy cynicism with regard to any notion that the learning of mathematics can be made quickly and dramatically easier. This long-standing attitude is typified by Euclid's oft-quoted reply to King Ptolemy when asked to simplify the learning of geometry, "there is no Royal road to geometry".

34. The doubts of mathematicians concerning the rather extravagant claims of some of the early promoters of programmed learning, resulted in their determination to try out the medium for themselves. As a result many very substantial projects have been carried through in the preparation and testing of mathematics programs. I shall mention a few of them.

35. In 1961, the School Mathematics Study Group inaugurated an experimental "Programmed Learning Project", designed to produce and test a variety of programs on a single clearly-defined course (beginning algebra) for which S.M.S.G. had already prepared and tested two different standard textbooks. This project has been fully reported. (See [17]; this report also contains a very useful *Manual for Programmers).*

The writing teams contained experienced mathematics teachers and university mathematicians, as well as psychologists who were experts in programming. Initially two programs were prepared, using the Skinner (constructed response) and Crowder (multiple choice) modes. It is interesting to note that a book of about 500 pages in standard textbook form, occupied over 1,700 pages when programmed in "constructed response" form, and over 2,300 pages in "multiple choice" form. A later "hybrid" form (using both of the above forms, together with some passages of standard text) occupied about 1,000 pages. This development of a hybrid form resulted from experiences during the project in the use of the older forms, especially the apparent need to overcome the tendency to boredom caused by the unrelieved use of a single form. Hybrid programming can be considered as an S.M.S.G. contribution to the art of programming. The results of the S.M.S.G. experiment are given in detail in the report cited. Roughly speaking, they suggest that (insofar as these things can be measured), a well prepared program can be a useful learning device, and that its effectiveness as compared with the use of a standard textbook is likely to be neither dramatically better nor dramatically worse, when used in a reasonably sensible way. Another conclusion was that the hybrid form was more effective than either of the "pure" forms, giving the students more confidence in the use of terminology as well as improved performance, and giving greater satisfaction to the program writers. Perhaps the most significant single conclusion to be drawn from the S.M.S.G. study is that programmed learning deserves, and needs, a great deal more investigation and experimentation, in order to determine the best ways of combining different modes (including possible new modes) in order to maintain the student's interest and challenge his ability, as well as to condition him to certain desired responses. There is, of course, much more of value in the report, which I strongly recommend to anyone seriously interested in the use of programmed instruction for the improvement of school mathematics.

36. Programmed learning has mushroomed enormously in its very short span of existence, and while it has not lived up to some of the claims of the early enthusiasts, there is little doubt that it has made (and will continue to make) a worthwhile contribution to our armoury of learning resources, and to our understanding of the learning process. A great deal of experimental work is now in progress, but much more will need to be done before the full potentialities of this medium are realised. But it is safe to conjecture that it will eventually establish itself as one of many useful resources, and not, as some of its early promoters saw it, as the solution to many of our problems of teacher supply and training. All the evidence to date suggests that programmed material will only be an aid to the mathematics teacher, and will not supplant him.

37. This is all that I wish to say in order to introduce this topic, which will, no doubt, be widely discussed in our working parties. But before going on, I should point out that programmed learning is strongly linked with computer-assisted learning, about which I shall have more to say later in this paper. I conclude by giving a short reading list of some of the many books and papers which are relevant to the discussion of programmed learning in mathematics. To this list should be added the references already given, and the very useful reference lists which they themselves contain.

Leedham, J., and Unwin, D. *Programmed Learning in the Schools.* London, Longmans, 1965.

Fincher, G.E., and Fillmer, H.T. "Programmed Instruction in Elementary Arithmetic", *Arithmetic Teacher,* January, 1965.

Smith, M.D. "Some Considerations in Teaching Mathematics by Programmed Instruction", *Mathematics Teacher,* May, 1962.

Kalin, R. "Some Guidelines for Selecting a Programmed Text in Mathematics", *Mathematics Teacher,* January, 1966.

Smith, L.W. "The Use and Abuse of Programmed Instruction", *Mathematics Teacher,* December, 1965.

Heimer, R.T. "Designs For Future Exploration in Programmed Instruction", *Mathematics Teacher,* February, 1966.

School Mathematics Study Group. *Programmed First Course in Algebra,* (Student's Text, Parts I & II, Student's Response Booklet; Teacher's Commentary). New Haven, Yale U.P., 1965.

Mathematical Association of America, Committee on Educational Media. *A Programmed Calculus.* New York, W.A. Benjamin, 1968.

Television and Films

38. It is convenient to consider these two media together. Anything that can be put on film can be televised (including colour film on colour television – at a price!); and any television program can be recorded on film and used through a normal film projector. However, there are a number of important differences in the use of these media. Films have the potential for far greater visual quality; but television has the advantage in flexibility of transmission, and in permitting widespread simultaneous use – sometimes by millions of students at the same time, as happened with the well-known "Continental Classroom" series in the United States.

39. It is likely that every country which has public television has used (or will use) this medium at some time or other for educational purposes, and that this will include the teaching of mathematics. A few years ago some educational administrators were enthusiastically hailing television as the future "solution" to the teacher supply problem, but this has not happened. As far as mathematics is concerned, there are severe limitations to the use of television. By far the greatest amount of communication in the mathematics classroom is by means of a mixture of visual and aural stimuli, with the visual predominating. Except in the very simplest situations it is necessary to have in view (or available for being viewed at a glance) more visual information than can be carried on a standard (25 inch or less) television screen. And the use of some of the very large screen television projection devices which are now available (at considerable cost) does not help, because of the nature of the television picture with its relatively small number of picture lines to the frame – usually in the neighbourhood of five hundred to seven hundred. When this image is enlarged up to theatre-screen size the line spacing is increased, and in order to smooth out these discontinuities the viewer must increase his distance from the screen. All that is gained is the possibility of having many people view the same screen at once, instead of having a large class grouped for viewing around a number of small screens.

40. It is well known that seemingly irrelevant psychological factors can be very important in the learning process. For example, many educators have suggested that the effective use of television for educational purposes is hampered by the fact that we are conditioned to its dominant use for entertainment. It would be interesting to see careful experimentation in the use of television for education, in societies that have not already been "contaminated" by its use for other purposes. This is not to suggest that the use of this medium for its present (largely entertainment and advertising) purposes might be curtailed, but if we are to try to understand the potentialities of television for education we might learn something from the study of the "extreme" situation in which it is not used for any other purpose.

41. Like many of you, we in Australia have experimented with the production and use of mathematics programs on television. It is possible by careful planning to adjust to the severe restriction on written visual material, but I do not know of any television teacher of mathematics who did not feel that this limitation hampered his effectiveness. The limitation becomes more serious as the level of instruction increases, and at the tertiary level (where one frequently covers many chalkboards with a single mathematical development, with frequent reference back to earlier parts of the development) the medium has very little value at all.

42. In considering the potentiality of television for mathematics instruction the limitation of writing space is the aspect which immediately strikes the mathematics teacher, and this is probably the single most important reason for the lack of enthusiasm which most mathematicians have for the medium. It is instructive to compare this with the enthusiasm of some of our scientist colleagues, and to remind ourselves of some of the reasons. Scientists like television for its ability to show what is happening in some inaccessible place; for its ability to magnify for the class some detail of an experiment, such as the behaviour of bacteria as viewed under a microscope; for its ability to provide a close-up picture in a potentially dangerous situation – as in chemistry, or in dealing with radioactive materials; for its ability to show what is happening in an environment which must be kept sterile – as in a surgical operation; and so on. But none of these special uses is relevant to mathematics: a mathematical derivation or formula is not dangerous; it does not need to be kept in a sterile environment; and it gains nothing in logical clarity by being magnified or viewed in close up. Thus not only does television "cramp the style" of a mathematics teacher, but there is no easy way in which the mathematics teacher – in contrast to the teacher of science – can significantly exploit many of the potentialities of the medium.

43. This is not to say that television has no place in the teaching of mathematics. Much elementary mathematics teaching is related to the real world. Demonstrations with concrete materials, relevant experiments, animation, and so on, can be recorded and used via television (just as they can via film) in order to supplement classroom experiences. When used for this purpose the television picture loss in quality, as compared with film, but it has the advantage of being able to show a situation in real time (i.e. while it is happening); and it has the advantage in the ease with which the number of simultaneous viewers can be multiplied indefinitely, using closed circuit or broadcast transmission.

44. A few years ago, in the State in which I live, a new mathematics program was being introduced in the secondary schools. With the technical assistance of the Australian Broadcasting Commission, our Mathematical Association produced a series of thirty half-hour television programs. Each of these was broadcast to schools on three separate occasions, so that the schools could choose their own viewing times. These programs were ostensibly directed at students. But it was the opinion of most of those who were associated with this project that its greatest value lay in the assistance which it gave to teachers, as indirect inservice training. A corresponding conclusion has been reached by others in similar situations.

45. Looking ahead, it seems safe to say that the usefulness of television for allowing large numbers of simultaneous users, for the exploitation of visual material other than mere writing or printing, and for teacher inservice training, together with the possibility of recording and re-using material, will result in a steadily increasing use of the medium as an aid to the learning of mathematics. It is to be hoped that this use will be both critical and selective, and that ways might be found to exploit the real potentialities of the medium – those things which can be better done by television than in any other way.

46. A useful overview of the use of television in mathematical education in the United States is given in a recent report[18] of a conference on "Television in Mathematics Education", sponsored by the National Center for School and College Television. After viewing much of the very considerable volume of recorded television material (for mathematics) which is available in the United States, it was the opinion of the viewing panel that negligible use had been made of the real potentialities of the medium and that, on present evidence, television should not be used as a basic resource for mathematics instruction, but only as a special purpose supplementary resource. But there is much more to this report, and I strongly recommend that you read it yourselves.

47. There is one further use of television that I wish to mention. This is as an aid in the pedagogic training of teachers. A couple of years ago, when at Stanford University, I saw relatively cheap videotape recording and playback equipment being used in connection with teacher training, in much the same way that tapes are used in a language laboratory. Videotape was used to record practice lessons, or parts of lessons. (The "students" were fellow trainees, but real classes could be used, with some modifications). These were then played back and discussed and criticised in a seminar situation. Playback could be immediate; it could be stopped at any point, repeated, and so on. This appealed to me as an immensely valuable tool for teacher training. Most of us are rather appalled at the quality of our own voice on a tape recorder. I am sure that we would be even more humbled by the pedagogic deficiencies which would be disclosed by videotape recordings of our class teaching.

48. With films, as with television, it is relatively easy to record a standard classroom situation, and this has been done extensively. Most of the offerings in the 1963 listing and review[19] of mathematics films, and in the 1967 listing[20] of broadcast television offerings in mathematics (both sponsored by the National Council of Teachers of Mathematics) are of this type. But this simple and direct approach makes no significant use of the real potential of the film medium, and a number of

projects have been undertaken in an effort to do better. The common experience of most of the groups which have attempted to come to grips with this problem is that it is enormously difficult and costly to produce a really satisfying product.

49. One of the most informative, and best documented[21] film projects is the so-called "Level I Project" of the Mathematical Association of America's Committee on Educational Media. The original objectives of this project were to prepare 50 to 60 forty-five minute films which would provide the major expository portion of an undergraduate college course in number systems, designed primarily for use in the pre-service training of elementary school teachers of mathematics. The films were to be accompanied by text materials, problem sets, sample examinations, programmed material, and so on, (a true multi-media approach), and to be supplemented by classroom discussions, problem sessions, and examinations. Every effort was to be made to "use the potential of the film medium for animation and special effects" and due attention was to be paid to "the importance of the individual teacher, whose personality and enthusiasm must shine out through the film in a way that it cannot in a text".

50. As the project developed, from the summer of 1964, the members began to appreciate the enormity of the task they had undertaken. The original scope of the project was significantly reduced, and it was only due to the dedicated efforts of a few individuals – such well known men as Carl Allendoerfer and Julius Hlavaty – that the reduced project was brought to completion. During the Conference I hope to show some of the films (which are available for purchase or for rental: see[22]) from this project.

51. The report of this project should be required reading for any group contemplating the production of mathematics films and film strips as part of a multimedia approach to the teaching of mathematics. While the diminished project was, in a sense, completed, the work done suggested many more lines of activity which could not be pursued because of limited resources. No attempt was made to determine the best mix of the media used (films, film strips, text, programmed material, live teacher) – indeed the report suggests that what is "best" is likely to vary from class to class, and even from student to student. As far as films are concerned, a major conclusion of the report is that the cost of making a polished production of a full series of long films (essentially as filmed lectures, supplemented with various props) would be very high, and that the money could be better spent in other ways. On the other hand, some value is seen in the use of shorter films for motivation, and as aids to the intuition; but the author of the report is careful to point out that this tentative conclusion has not been carefully and objectively evaluated. Animation is seen as highly suitable for most mathematics films; but films of "stand-up lectures" are seen as largely ineffective for the learning of mathematics, although they might be of great value for other – e.g. archival – reasons. (Consider the value of a film showing Gauss or Newton – or even, alas, Einstein – as a teacher).

52. The M.A.A. Project is a pioneer in its attempt to exploit the full potentialities of the film medium, as part of a multi-media approach. There is no doubt that much more will be heard about this type of approach; but it is already clear

that a great deal of work will have to be done before we are in a position to understand and exploit it fully.

Radio

53. A few months ago I was aware of only two recent projects concerning the use of radio as a resource for learning mathematics. The first (which is run by the University of New South Wales, using its campus radio station) has two components: one of these is a remedial course designed to bridge some gaps in the mathematical education of students who are about to enter the University; the other is an in-service course for teachers. The broadcasts are keyed to written materials (which are sent out in advance to enrolled students), and it is not likely that they have any value for those not in possession of the written materials. My colleagues in Sydney seem to be reasonably satisfied that these programs are useful for the special purposes for which they are planned: that is, to reach a large audience at very low cost; and to give them a somewhat more personal stimulus than they would obtain from using written materials alone. Except for the difference in cost, this two-media approach (radio and keyed text) would seem to have no advantages over television, and this is, no doubt, the reason why this sort of program has received very little attention.

54. A second example of a project which uses radio and supplementary written materials, is the fairly recent introduction, in Australia, of "Schools of the Air". In the state of Western Australia about 500,000 square miles (roughly half of the area of the state) is covered by broadcasts from four transmitting school centres. These broadcasts are designed to assist the correspondence school education of children who live in such isolation that there are not enough of them living in an area of many thousand square miles to justify the establishment of even a one-teacher primary school. In addition to the correspondence work (which has been operated by the State Education Department over a long period of time), "School of the Air" broadcasts (using 2-way radio transceivers) were begun in 1959, in conjunction with the use of the same radio equipment as part of the Royal Flying Doctor Service. These broadcasts, which run for 2½ hours on each school day, cover the whole of the primary school program, including mathematics. Their two-way character is an important part of the project, permitting a significant amount of direct "feed back" from student to teacher. As the number of students at any one point (usually a cattle station) is generally less than five, the per student capital cost of the equipment is relatively high. But the cost is partly justified by the parallel use of the equipment for medical services. The whole operation covers about 200 students spread over seven grades. It involves about 10 teachers, who also handle the correspondence work. Some of the other Australian states have similar projects.

55. Just recently I came across another report[23] of an experiment in the use of radio as an aid in the teaching of mathematics in Wisconsin, U.S.A. This experiment consisted of a fifteen-minute broadcast, not keyed to any particular written material (i.e. it was self-contained), and neither the topic nor the script was available in advance. The experimenters reached the conclusion (based on replies to a questionnaire) that this procedure could be a useful supplement to the regular in-class teaching program. It is difficult to see that radio, as such, played any significant

role in the experiment; possibly the same result could have been achieved by the use of a tape or phonograph recording.

Structured Aids, Analogy Devices and other Classroom Materials

56. The materials which come under this heading are, presumably, covered by the "etc." in my terms of reference. Their relevance is greatest at (but by no means confined to) the primary level, and for this reason I have not had any significant involvement with any of them, except, of course, those which we also use extensively at the tertiary level. (e.g. chalkboards, computers and other calculating devices, overhead projectors). I am sure that many of you will be much more familiar than I am with the ever-growing variety of structured aids now available. Cuisenaire rods, Stern materials, attribute blocks, Dienes' multibase arithmetic blocks, pegboards, number balances, mirror cards, discovery boards, geoboards, abaci, slide rules, desk calculators, and so on. Many of these devices have their origin in the imaginative ideas of gifted teachers, and in the discoveries and theories of child psychologists concerning the learning processes of children. In addition to the well known (and commercially available) devices, there are undoubtedly thousands of home made and readily available materials which are being exploited by imaginative teachers as aids in the learning of mathematics. The Nuffield Mathematics Project[12] places great emphasis on the use of commonly available resources in the school and in the community: a great many of the "mathematics tasks" designed by its teachers for the early (and for the not-so-early) school years, involve the handling of physical materials. Many of the leaders in mathematics education for the primary grades (e.g. Robert Davis[4]) believe that the Nuffield Project is carving out a path which will be increasingly followed in the future. This is not really surprising if we believe that the environment in which we live has present significance in relation to personal mathematical development, as well as its well known historical role in the development of mathematics.

57. It seems safe to predict that, along with an increasing use of the environment (possibly through integrated science and mathematics programs), we will see an increasing use of specific, carefully devised, analogy-devices, such as the Stern, Cuisenaire, and Dienes materials, and that the most effective teaching will make use of all of these and the many other teaching aids which are, or will become, available. It is most unlikely that the path to success will lie in the exclusive use of any single device.

58. I conclude this section with a short reading list of some of the better known books on the use of concrete materials:

Stern, C. *Children Discover Arithmetic.* London, Harrap, 1958.
Stern, C. *Experimenting with Numbers.* New York, Houghton Mifflin, 1954.
Cuisenaire, G., & Gattegno, C. *Numbers in Colour.* London, Heinemann, 1954.
Dienes, Z.P. *Building Up Mathematics.* London, Hutchinson, 1960.
Dienes, Z.P. *The Power of Mathematics.* London, Hutchinson, 1963.
Dienes, Z.P. *An Experimental Study of Mathematics Learning.* London, Hutchinson, 1963.

Dienes, Z.P. *Mathematics in the Primary School.* Melbourne, Macmillan, 1964.
Sealey, L.G.W. *The Creative Use of Mathematics in the Junior School.* Oxford, Blackwell, 1960.

Computer-Assisted Learning

59. No discussion of the resources for learning mathematics can ignore the exciting new developments of the last few years in the use of computers as learning aids. This is the second way in which computers are going to have a major impact on mathematics education. (The first, which will undoubtedly be discussed in other plenary sessions, is the impact of the computer on the curriculum itself).

60. Among the best known experimental projects which use digital computers as teaching aids are those of Dr. Patrick Suppes, at Stanford; and Project Plato, at the University of Illinois. (See Reference [4]). Roughly speaking, the computer is used as a programmed learning device. A large time-shared computer can be programmed to give a degree of flexibility which would require many thousands of pages of printed matter, and which would be forbiddingly difficult to follow. The computer can make virtually instantaneous shifts to different parts of the program, guided by the most recent answer which the student gives (or even by a large number of previous answers). Moreover the growing capacity of computers makes it possible to envisage (as in Project Plato) as many as 6,000 simultaneous users (located in schools, universities, or even in their own homes and offices) of the one program on the same computer. And while doing all this the computer can store complete records of the responses of every individual, and print out these (or any pre-determined analysis of them) on command.

61. Thus the computer holds out the prospect of providing a vastly greater individualisation of teaching, limited only by the capacity of the computer and by the imagination and skill of the teacher-programmers. Moreover it promises to give the teacher an almost unlimited capacity for keeping records in immediately accessible form on each individual student. While it is not likely that any foreseeable computer will achieve the flexibility of response of a superior teacher who has only one student, computer-assisted instruction holds out great promise for significant progress in the direction of tailoring the education of each individual student to suit his own capabilities and his personal requirements; and, properly programmed, it can even assist in determining these capacities and needs.

62. At the present time the work being done is almost entirely experimental, and the cost is high. But this is falling steadily with technological advances, and there is little doubt that computer-assisted learning will be increasingly used in our educational systems. At least, as Robert Davis has pointed out, this seems to be the view of several of the large American companies with computer interests, many of whom have recently bought, or merged with, book publishers and manufacturers of other educational materials.

Evaluation

63. Evaluation of the effectiveness of any educational program, curriculum or learning resource is enormously difficult, and yet enormously important. Probably

no one has been more aware of the need for some measures of effectiveness than Dr. E.G. Begle, Director of the School Mathematics Study Group. Yet in a recent issue (March, 1968) of *The Mathematics Teacher,* in reference (inter alia) to investigations by his own group of the effectiveness of a programmed algebra course, he writes "None of the above can be considered to be properly designed and executed experiments ... important variables were either ignored or left unmeasured ... control over treatments was not always feasible ... the purpose of the studies was to obtain rough estimates... ."

64. It is safe to say that the same comments could be made about virtually every effort which has been made to evaluate the effectiveness of new curricula, or the usefulness of the growing number of teaching aids and other learning resources. Some idea of the difficulty and the cost of a fairly well designed and well executed experimental program can be gauged from two recent major studies concerned with mathematics education. One of these is the twelve-country UNESCO-sponsored "International Study of Achievement in Mathematics", whose two volume report[24] is notable for what it discloses about the difficulties and shortcomings of such a venture, and for the modesty of its claims concerning what has been "proved". The other is the National Longitudinal Study of Mathematical Abilities, a project of the School Mathematics Study Group. This was designed (in 1962) as a five year study to assess the effectiveness of new mathematics curricula. The report of this massive project (involving well over 100,000 students, from some 1,300 schools, at a cost of millions of dollars) is awaited with great interest.

65. The lesson to be learnt from these major projects is *not* that evaluation is so difficult that we should not bother to attempt it. It is essential that those using new learning resources should adopt an experimental and critical attitude, and that they should attempt to form some idea of the effectiveness of the resource for them and for their own students; but at the same time they should be aware of the largely subjective character, and the limitations, of present methods of evaluation. In view of these limitations, it is particularly important to beware of those educational conservatives who refuse to participate in any kind of innovation until its effectiveness has been "proved". It is safe to say that absolute "proof" will never be possible, and yet innovations must be made, guided by the collective wisdom and experience of mathematicians, teachers and educational administrators.

Summing up

66. What does this all add up to? It seems to me that there will be a growing use of physical and other "real world" aids to instruction (both in the classroom and outside it) at the elementary level, and that this will be more closely associated than it has been with experiences in "science". This will not supplant the current and growing emphasis on "structure", but rather it will provide the concrete experiences from which to abstract, and it will provide that evidence of "usefulness" which most children seem to desire. At all levels this will be increasingly supported by a growing sophistication in the design of textbooks and programmed materials, and in the use of new and existing communications media of all types – teachers, books, overhead-projector transparencies, films and film strips, television, computers etc. I do not see any particular resource or medium as predominating, but rather that there will

be a wider and more carefully planned multi-resource and multi-media approach – the kind of approach that good teachers have always used with the limited resources and media available to them. Perhaps in some distant generation, when we might have reached agreement on the aims of mathematical education (individualised to the need, the ability and the personality of each student), and when we might have mastered the problems of measuring the effectiveness of different curricula, resources, and media in achieving these aims, we will be able to put our computers to work to devise and control the optimum educational mix for each student. But this possibility is so remote that we should not let it spoil our pleasant anticipation of many more Conferences such as this.

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REPORT OF WORKING GROUP B.3

Chairman: Professor R.S. Varma (India)

67. Resources for learning mathematics as discussed by Group B.3 are presented in the following order -

(a) Books
(b) Audio-Visual Aids -- Films, Film Strips, T.V., Radio, etc.
(c) Programmed Learning
(d) Low Cost Teaching Aids and Demonstration Equipment
(e) Resources involved in Mathematical Education at different levels
(f) Some Additional Resources

Books

68. Some criticism was expressed as to the type and age of some of the textbooks in use in some developing countries. The general feeling was that within a region attempts should be made to use and if necessary to write textbooks written along modern lines appropriate to their needs, and it was pleasing to hear reports from some areas that this was already in hand, and was being successfully pursued. It was thought that each country should proceed as it thought fit, seeking help from the developed countries if necessary. There might well be a need for international co-operation, for example, in the dissemination of knowledge about available books and it was considered desirable to put on tour collections of new books which could be circulated from country to country and even within a country itself.

69. Members emphasized the need for varying categories of books:-

(i) Books meant for pupils. These could be either textbooks, or small topic books, or books of reference to which students could refer, as well as attractive reading books on mathematical topics.
(ii) There is need for books of reference for teachers both on the purely mathematical front and on the teaching front which would suggest such things as: (a) modern teaching – approaches, (b) suggestions for the making and using of apparatus.

The view was expressed that there was a place for books on modern methods of teaching mathematics as well as modern content.

70. The need for background books which would enrich the mathematical imagination of pupils and teachers was emphasized Books of this kind suitable for pupils below the standard of the ordinary level examination were now available.

Audio Visual Aids – Film, Film Strips, Television, Radio

71. The Oversea Visual Aids Centre (O.V.A.C.) Tavistock House, South Tavistock Square, London, W.C.1., was established some five years ago in order to disseminate

information about visual aids, their consideration and use. It publishes a bulletin, details of which can be had from the above address.

72. The use of slides, colour or otherwise, was discussed, but it was felt that in the context of mathematics teaching, a slide projector was of very limited use, and was in fact being superseded by the overhead projector. The advantages of the overhead projector were said to be:-

(a) That not only could straight pictures be put on, but a sequence of transparencies could be used to build up a composite picture as a lesson proceeds.
(b) The structure of such overhead projectors makes it possible for shadows of solid objects to be cast on the screen. In the teaching of motion geometry this would prove a very valuable visual aid.

73. The role of Television in mathematics teaching was discussed at some length. In this respect television is still evolving and this medium has not yet been fully exploited. That television certainly cannot replace a teacher but can help to compensate for the shortage of *qualified teachers* was emphasized.

Television can be used for the following purposes:

(a) To explore regions of mathematics which lend themselves *(or could lend themselves)* to visual interpretation, especially of a type that teachers cannot easily cope with for one reason or another. The means to this end are very varied but usually involve film animation, electronic wizardry, models and animated captions.
(b) To explore fresh areas of mathematics with which most pupils and many teachers might not be familiar. This often means re-interpreting rather difficult books and inviting university and college lecturers to the studio. Occasionally it involves extensive actuality filming (e.g. of computers in operation).
(c) Linked with (b) is the task to explore the mathematics all around by drawing on examples from everyday life and by showing applications in science, technology, sociology, etc. Apart from the techniques already mentioned, photographs are often of value here.
(d) To use the medium to "pipe" all kinds of mathematics to schools which are short of qualified staff. This can allow more effective use of the time of qualified teachers either by themselves or in some form of team-teaching.
(e) To encourage and advance the work of slow learning pupils. Programmes produced by the B.B.C. for slow readers using the full gamut of visual techniques have been particularly successful and it was thought that this might provide a parallel for the learning of mathematics by the slow learning pupils. One of the most surprisingly successful devices used for encouraging children's work at all levels of ability has been that of showing films of other children working.
(f) To inform teachers of developments in the subject matter and teaching of the subject using all the techniques mentioned above.
(g) To act as a source of secondary material in the form of books, films, etc.

74. Possible uses of video tape to record and supplement transmitted programmes were also considered.

75. In some areas the programmes were supplemented by correspondence courses based on the programme material.

76. This Committee could find very little experience of radio being used as an agency for mathematics teaching, but it was posed as a possibility that it might well be used in the context of the in-service training of teachers for critical discussions of television or other programmes.

77. It was felt that one of the advantages of using television and radio, was that the wider audience of parents and other interested parties could be reached as well as the pupils and teachers to whom the programme was really directed. This was considered a great advantage when considering any form of educational change whether of mathematical content or of approach.

78. Some interesting experiments were reported by way of two-way telephonic links sometimes supported by ingenious electronic visual devices, but it was felt that these had not yet realised their full potential and were very experimental in nature.

Programmed Learning

79. It was pointed out that the ideas underlying programmed learning had been a valuable teaching technique for many years. What, however, was new, was an influx of hardware and/or books which were specifically tightly programmed. The view was expressed that most of these approaches were often deadly dull in their impact on pupils, but nevertheless they may find a positive use particularly in remedial situations, e.g. to remedy a gap in learning due to absence or some such other cause. The point was made that no machine is better than the programme which is fed into it and it was felt that many of the commercial programmes did not measure up to our requirements. It was felt that at best such machines would facilitate the learning and practice of skills rather than the understanding of concepts.

Low Cost Teaching Aids and Demonstration Equipment

80. With regard to the production of low cost teaching aids, the first matter considered was what should be produced, and it was pointed out that mathematical teaching journals in the developed countries often contained articles and references about making and using such aids and demonstration equipment; for example, Mathematics Teaching No.18, Journal of the Association of Teachers of Mathematics (U.K.) is devoted to this. A further source would be the commercial catalogues from school equipment firms. It was felt that Ministries concerned in the various countries should encourage the local production of such teaching aids and equipment by whatever local means are found to be appropriate in their circumstances.

81. Some suggestions (minimal) with regards to equipment were made:-

(1) *Squared Paper:* Various problems would demand squared paper of different unit squares and, therefore, a *variety* of squared papers should be available.

A blackboard or other board ruled in squares to be used for demonstration purposes should also be available.

(2) *Physical Apparatus:* It was felt that some form of co-ordinates board was a very useful feature. Nails at the corners of congruent squares covering a board form a pin board (peg board, geo board, lattice board) which can be used throughout all school grades.

(3) *Squares and Cubes:* A large collection of squares and of cubes for use by pupils was essential. Some of these should be unit squares, e.g. square inch; and unit cubes, e.g. cubic inch.

(4) *Fraction-boards:* Various forms of fraction-boards for developing the concept of fractional parts.

This list must be regarded as an absolute minimum.

82. The items are capable of being made locally from existing materials and are, therefore, low in cost.

Resources Involved in Mathematical Education at Different Levels

83. It was pointed out that in the United Kingdom a project was in hand to study ways and means of organising work in schools so as to make the best use of teachers' skills and new developments in method and equipment. The project was established in 1966 and is financed by Nuffield Foundation in co-operation with the Schools Council. Further information can be obtained from Mr. T. McMullen, Tavistock House, South Tavistock Square, London, W.C.1.

The group felt the necessity to state priorities in matters of resources:

Primary School: the priorities were thought to be:-

(1) Qualified Teachers.

(2) Materials arising from environment. This variously could be sea shells, mangoes, etc.

(3) Books (text, reading, reference).

Secondary School: the priorities were ordered rather differently as:

(1) Qualified Teachers.

(2) Books (text, reading, reference).

(3) Physical Equipment.

Some Additional Resources

84. Buildings, desk calculators and computer assisted learning were mentioned, but could not be discussed for want of time.

85. The group was unanimous that the most important resource for teaching was the teacher.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP B.3

Comments and Survey

86. As was pointed out by Professor Varma when presenting the report to the plenary session, so vast a topic resulted in the discussion covering only part of the field.

Resources could be seen to serve Primary Education, Secondary Education and Teacher Training. References to specific resources were contained in the reports of groups A.1, A.2 and B.2. This to some extent helped to narrow the discussions undertaken.

87. Attention was drawn to Modern Curriculum Developments in Britain published by CREDO in 1968. This was regarded as a useful source book for those seeking details of British educational projects, mathematical and otherwise.

88. Educational technology is advancing at a pace comparable to other technological developments. It was in the spirit of "What is new and strange today becomes an everyday affair very quickly", that delegates heard with interest of the "hardware", being introduced in some countries. For various reasons, including the economy of the country, the lack of electric power and the depredations of some tropical insects, many delegates felt that any immediate application of much of the "hardware" was not for them. It was, however, noted that in some developing countries, television had been introduced as an educational medium, and that several countries had instituted centres for audio-visual aids. It was not so much in regard to the media themselves but in the use made of them that fears were expressed. These fears could be placed in two classes:-

(i) That available material, either good or bad, suitable or unsuitable, would be uncritically accepted in the schools. Such uncritical acceptance would not be in the best interests either of the pupils or the teachers. The same can of course be said for the manner in which some textbooks are used or misused.

(ii) That the radio or television voice would be regarded as "the voice of authority", and some teachers would tend to sit back and "let the machine do the work". The teacher being replaced by the machine was an often expressed fear; but as one delegate remarked "The teacher who can be replaced by a machine deserves to be replaced by a machine".

The remedies for these doubts would appear to be:-

(i) That teachers themselves control the content of the programmes. This will later be discussed at greater length when considering television.

(ii) That by initial and in-service training, teachers can be helped to use modern devices wisely as productive tools of their trade rather than as ends in themselves.

It was thought that in mathematics teaching a multi-media approach was the desirable one; it was for each country to find ways of making optimum use of all the resources available to it.

A detailed historical account of developments in the U.S.A. will be found in *A History of Instructional Technology*, Paul Seitler, McGraw Hill (1968).

Note was made of the existence in the United Kingdom of the *National Council of Educational Technology* operating from 160 Great Portland Street, London, W.1. One of its functions is to bridge whatever gap there may be between the users and the makers of such teaching aids.

89. *Desk Calculators* were regarded as desirable but expensive. As calculating devices, many felt that it would be a very long time before they would supersede the slide-rule or the logarithm table. More hope was held out for them as teaching aids in the primary and lower secondary school. It was claimed that as well as motivating learning, the pupils were helped in grasping the concept of place value; that the relationships between addition and subtraction, addition and multiplication, subtraction and division were clarified, in that they were operative procedures on the machine.

90. *Computers and Computer-assisted Learning.* Where computer studies had been introduced the interest of the students was phenomenal. Computer science experiments were reported from Ontario, Wales and England. In the case of the Welsh experiment, a telex link had been made between the school and a university computer. The aim of the operation was twofold:-

(i) Using the computer as a computing tool at the VI form (6th and 7th year secondary) level.
(ii) Helping pupils of all ability ranges to understand the applications, implications and limitations of computers rather than to lead pupils toward computer programming.

It was thought that work on computer assisted learning was not yet sufficiently developed for its worth to be properly evaluated. Some fears were expressed that we might revert to the sort of teaching and learning situation from which we are now struggling to escape. Care would be needed lest a straitjacket of one pattern be replaced by a straitjacket of another pattern. Even so, it was felt that the method might have considerable potential for certain aspects of mathematics.

91. *Television* for schools has two separate connotations:-

(a) Programmes broadcast on open national networks.
(b) Closed circuit television within a college or school, a group of schools, or perhaps throughout the schools of a particular area. A number of teacher training colleges in Britain are equipped with closed circuit television, as are schools in Glasgow, Leicester, London and Plymouth, to name a few authorities using this medium.

In the second case the responsibility for content, programme making, production and transmission are entirely in the hands of the teachers and educational advisers concerned. Such a set up should allay the fears of those who are critical of programmes being imposed from outside the schools.

On the national networks in the United Kingdom, the general policy for school programmes and the scope and purpose of each series of programmes are laid down by councils on which sit representatives of the professional associations of teachers, local education authorities, the Department of Education and Science and other educational organisations. Production of programmes and transmission are matters for the broadcasting authorities. Practising teachers often present the programmes. The teacher links here are again strong.

The conference background paper *The Role of Television in Mathematical Education* gave an exposition in some detail of the current situation in Britain. It also discussed some detail of the *Maths Today* series of the B.B.C. The 20 minute programmes are aimed at first and second year secondary pupils and each programme is transmitted at four different times in a fortnight; side by side with these programmes for the pupils are fortnightly supporting programmes for teachers in the *Teaching Maths Today* series. This work is further supported by the written word in the form of guides for the teachers and work/study sheets for the pupils. All this can be augmented by 8 mm. film loops in cassettes which are on sale. Evaluation of such a multi-media approach has yet to be made.

It would be difficult either for teacher or pupil to be a non-participant in such a widely based operation. Participation was perhaps even easier in a Scottish experiment, reported by a delegate where the television programmes consisted of direct teaching to pupils. The programmes were interrupted at appropriate points – at a question perhaps, or a point for brief discussion with the teacher in the room. During these short breaks the teacher took over, and then the programme proceeded to the next phase. The delegate continued "What these programmes have done to the teachers has been remarkable – it is perhaps the best training device I have ever seen. What it has done for the children I shall know next year".

92. *Film loops* were thought to have considerable potential in that there was flexibility in the selection to be made and they were less expensive than long films. Furthermore, individual work assignments at varying levels could be based on a suitable film loop. Film loops were thought more useful than a normal film, which was considered to be better from the background or enrichment point of view than from the direct learning viewpoint.

93. It was natural that the newer and perhaps therefore more exciting forms of educational tool should, in such free discussion, take precedence over the more traditional, humbler, less expensive tools such as books, blackboards, environmental and other physical materials. In the realistic spirit of the conference, it was felt that for many countries, ways and means of making optimum use of the resources currently available to them should continue to exercise all their powers. The lead paper and report under discussion rightly emphasized these humbler features, but perhaps equally rightly the discussion complemented this by dealing with the more novel resources.

Since learning is an individual affair, any artefact which will stimulate and ensure sound individual learning is worthy of serious consideration by all educators.

CHAPTER IX

Commonwealth Co-operation in Education

1. National boundaries have never been iron curtains for education. The cross-fertilization of educational ideas has been a feature of scholarship over the centuries but whereas in medieval times most of this cross-fertilization occured through the involuntary journeys which scholars often found themselves compelled to take for the preservation of knowledge, the last few decades have seen developed a conscious and planned effort at international co-operation.

UNESCO's work in the field of mathematics

2. The major instrument now spearheading this development at the international level is the U.N. Educational, Scientific and Cultural Organisation. The Conference in Trinidad was fortunate in having as an observer from UNESCO Professor G. Soos of its Division of Science Teaching in Paris. With Professor Soos was associated Mr. H.L. Rudstrom of the UNESCO Teacher Training Project in Jamaica. Within UNESCO, mathematics is one of the subjects handled in the Division of Science Education. At a session of the Conference devoted to Commonwealth Co-operation Professor Soos gave an account of UNESCO's work in this field. This has included the collection and exchange of information with member states, the publication of surveys and new trends, the design and implementation of experimental programmes in different regions of the world, and programmes of lectures, fellowships, etc. Current projects in mathematics include a postgraduate course on Topology and the Foundations of Mathematics in Poland, a Functional Analysis course in Denmark, a course on Probability and Statistics in Budapest and a European meeting of mathematicians in Bucharest to tackle some problems of mathematics education at the secondary school and university levels. The work of the Organisation in Africa, in Asia and in Latin America in teacher-training and curriculum development involves improving mathematics teaching. In his address to the Conference Professor Soos described a UNESCO project for the improvement of mathematics instruction in the secondary field in countries of the Middle East, launched a year ago. For this project a study group for the modernisation of mathematical education has been set up in each of the eight participating countries.

Bilateral aid within the Commonwealth

3. Commonwealth co-operation in the development of improved methods and new ideas in the teaching of mathematics has up to the present been mainly bilateral in character. Some of these bilateral arrangements have proved a major factor in developing new programmes and projects.

At the Third Commonwealth Education Conference at Ottawa in 1964 (which had recommended a Commonwealth Conference on the teaching of mathematics) the British delegation indicated that Britain would be prepared, if the conference so wished, to consider securing the services of a team or teams of experts in curriculum planning and research who could make their advice available to other Commonwealth countries on request. It was as a result of the interest shown by other countries in participation in this kind of programme that in October 1966 that Britain set up in London the Centre for Curriculum Research and Educational Development Overseas (CREDO). Reference to the work of CREDO is made in the section of this chapter which deals with the British programme. The Fourth Commonwealth Education Conference at Lagos in 1968, in reviewing these developments and the work of curriculum units in other Commonwealth countries, drew attention to the need for more interchange of information about them. It also emphasized the importance for education generally, that the reasons for success or failure of projects or experiments in curriculum development should be assessed and conveyed to all member countries. The Lagos Conference clearly envisaged the Education Division of the Commonwealth Secretariat being strengthened to enable it to play a more effective role in aiding programmes of curriculum development and in working more closely with national organizations in this field. Summaries of the bilateral programmes of the developed countries of the Commonwealth who were represented at the Conference follow.

Britain

4. The assistance Britain gives to the development of mathematical education in other Commonwealth countries takes two main forms: the provision of information and expert knowledge through the supply of books and equipment, the interchange of personnel etc., and help with training and teaching by means of short or long-term exchange visits.

5. The major agency involved in the provision of assistance is the Ministry of Overseas Development but supplementing the Ministry's efforts are those of the British Council, the Centre for Curriculum Research and Educational Development Overseas (CREDO), the Centre of Educational Television Overseas (CETO), the Oversea Visual Aids Centre (OVAC), the Inter-University Council, the Association of Commonwealth Universities, the Nuffield Foundation, British publishers and many other private and professional bodies.

Teachers and Teacher Training

6. In terms of finance and the number of people involved, a large proportion of Britain's contribution to the development of mathematical education in other Commonwealth countries is the provision of school teachers and lecturers in Colleges and Departments of Education. This kind of help has been given by Britain for many years but the conditions of work and the terms which can be offered have radically changed. Nearly all those now appointed are on a contract basis, many of them for relatively short periods of two or three years in the first instance.

7. In 1967 the Ministry of Overseas Development made 1459 appointments in education, some 70 per cent of these being concerned with English, mathematics and science, and probably about 20 per cent being concerned in one way or another with mathematics.

Voluntary Service Overseas

8. The programme of service overseas by graduate volunteers and school leavers continues to be operated by the voluntary societies with the financial backing of the Ministry of Overseas Development. The growth of the scheme since 1962 has been impressive. Under the 1962/63 programme 320 volunteers, of whom only 32 were graduates, worked in 52 countries. By the end of 1967 almost 2,000 volunteers, of whom about two-thirds were teachers, were serving in 75 countries. Stress is now being placed on the recruitment of volunteers for longer periods. It is realized that any scheme employing young and inexperienced teachers is bound to have its drawbacks, but nevertheless, with mathematics teachers of any kind in such short supply, the volunteer teacher can be of great assistance and there are instances of the gifted and enthusiastic volunteer playing a major role in the establishment of lively centres of mathematical reform.

Serving Teachers

9. One important aspect of teacher training is that given to teachers already in service. Systematic help with teacher vacation courses overseas was instituted in 1961, and in 1967 over 130 teachers from British colleges of education and schools conducted courses in 20 countries for some 1,700 local teachers under the Ministry of Overseas Development's Teacher Vacation Course scheme. Courses on secondary mathematics were held in Ceylon, Kenya, Swaziland and Uganda and on primary mathematics in British Honduras, the Gambia, Lesotho, Nigeria, Swaziland, Tanzania, Uganda and Zambia. In addition to the assistance provided by the Ministry, the British Council and CREDO provided tutors for mathematics courses in Ghana, Tanzania, Malawi and Mauritius. In 1968 for the first time Britain has supplied tutors to an Indian Summer Institute in mathematics.

10. In many countries these teacher vacation courses are now established annual events which can play a most effective part in the introduction of new syllabuses and teaching methods into schools.

Schemes for Study in Britain

11. Assistance of this type, however, must be viewed as a stop-gap until indigenous mathematics teachers can take over the posts at present filled by expatriates. A major contribution to that desired end is made by those schemes designed to encourage study and training in Britain. Over 3,000 students and trainees are brought to Britain annually by the Minstry of Overseas Development. Of these some 600 are Commonwealth Scholars or Teacher Training Bursars who are financed under the Commonwealth Education Co-operation Scheme, the remainder being financed under Regional Technical Assistance Programmes. The training given covers many aspects in addition to education, but courses of special interest to mathematicians are those provided at the Universities of Hull and Southampton (and that which will shortly be provided by the University of Leeds) for Commonwealth Bursars. The value of this scheme is amply demonstrated by the number of former bursars who are now holding positions of great responsibility as inspectors, heads of department in teacher-training colleges, and headmasters. Those bursars from primary schools who are not specialising in mathematics are also given some insight into current developments in primary school mathematics, and an opportunity for bursars to meet some of the

leading personalities involved in the revision of the primary school curriculum was provided at a CREDO-sponsored vacation course held in March 1968.

12. Training of a more specialised nature is provided for overseas students at the Oversea Visual Aids Centre and the Centre for Educational Television Overseas, which, in addition to undertaking pioneering work in the production of educational programmes designed for use overseas (its algebra kit has been purchased by Ghana, Nigeria, Singapore and Uganda, and also by non-Commonwealth countries), also gives two courses annually.

The Exchange of Expert Knowledge and Information

13. The development of the curriculum, of improved teaching techniques and of methods of assessment must continue and must increasingly become a matter of international concern. Indeed, evidence of this concern is provided by the recent establishment of an International Journal of Mathematical Education. At the moment attention is naturally focussed on those countries in which the current wave of reform was initiated, but more countries are now carrying out experiments in mathematical education and it is only reasonable to assume therefore that the traffic in ideas will become more reciprocal.

The Work of CREDO

14. It was partly this demand from overseas countries for information about what was happening in Britain that led to the founding of the Centre for Curriculum Research and Educational Development Overseas (CREDO) which was established in October 1966. CREDO has already been able to provide several types of assistance in the field of mathematics education. For instance, it has arranged visits by British experts to other Commonwealth countries to work alongside developers there. For example, Dr. Geoffrey Matthews of the Nuffield Mathematics Project spent several weeks in Ceylon in Spring 1968. It has also provided tutors for teachers' workshops (Zambia, Guyana, Malawi, etc.), presented sets of texts and teachers guides to schools and training colleges (for example, in Uganda and Tanzania), given financial support to a local project (The Joint Schools Project of the Mathematical Association of Ghana), helped provide apparatus for use in primary schools (Mauritius), made it possible for teachers from Lesotho, Botswana and Swaziland to attend a teachers' course in Malawi and arranged to have draft textbooks and syllabuses criticised by experts.

A Co-operative Scheme in Ghana

15. How various agencies can unite in a large-scale development plan is exemplified by that recently begun in Ghana. In 1967 the Ghana Ministry of Education established a unit for curriculum development and research. This was done after it had received the report of a consultant, whose visit to Ghana had been jointly sponsored by CREDO and the British Council, and had had further talks with a representative of Canada whose Government had also promised assistance. The development of primary mathematics is only one part of the overall plan but it is interesting to see how this development is proceeding. For some years now the Ministry of Education has organised a three week vacation course for some of its teacher training college tutors, local education officers and primary school headmasters. At each of

these workshops two tutors have been provided by the British Council and in an attempt to secure continuity and a uniformity of approach these have recently been drawn from Leicestershire. These workshops provided a bridgehead and now the work is being extended. Four colleges are to take part in a pilot experiment to reform teacher training and a large scale attempt is being made to give additional training to the local education officers who will obviously have a key role to play in any developments in the primary schools. Special arrangements have been made for ten of these officers to come to Britain annually for the next three years. The officers, whose visit will be financed under the Commonwealth Education Co-operation Scheme, will attend a special course at the University of Hull.

16. These are some examples of the projects which are helping to make the teaching of mathematics more effective. Some of the major projects in Britain for reform in this field, such as the Southampton Mathematics Project, the Midlands Mathematics Experiment and the Nuffield Primary Mathematics Project, have attracted world wide attention and in their own ways contributed to co-operation.

Canada

Responsive Assistance by Canada

17. Education programmes financed by Canadian development assistance are responsive in nature, that is to say educational assistance is only given in response to a specific request for such assistance. The types of persons serving abroad under this programme include Advisers to Ministries of Education, Teacher Trainers, Teachers at the senior grades (forms) of Secondary Schools, Technical and Vocational Teachers and University Professors.

The Extent of Aid from Canada

18. In September 1968 approximately 700 Canadian educationists were serving overseas in 46 countries. Of this number 374 were serving in English-speaking countries of Africa, South-East Asia and the Caribbean while 329 were serving in Francophone countries of Africa and South-East Asia.

19. In the programme for English-speaking countries 87 teachers and 17 professors were serving in the West Indies and Guyana under the Commonwealth Caribbean Assistance Programme. Serving under the Colombo Plan, in South and South-East Asia, were 27 teachers, 5 technical and vocational instructors and 7 professors. In the special Commonwealth Africa Aid Programme there were 173 teachers, 20 technical and vocational instructors and 37 professors serving in East and West Africa. There is also one teacher in Western Samoa.

20. In the French-speaking programme area there were 249 teachers, 20 teacher trainers and 44 professors serving in the Francophone countries of Africa; in Cambodia and Laos, there were 9 teachers, 3 teacher trainers and one professor currently serving in the area.

21. Before 1960 only a small number of educationists were assigned abroad under the Government of Canada's programmes of educational assistance. That year, however, marked the beginning of the current comprehensive programme through which teachers, university professors and educational advisers are provided for the

developing countries. The increasing number of requests for educationists received from the Governments of these countries is the reflection of the priority that they give to the expansion and improvement of their educational facilities in the light of the critical importance to economic growth of adequate supplies of trained manpower.

The Project Approach to Aid

22. In the past teachers have been sent to the field on individual assignments in response to specific requests for educationists at all levels but particularly in the fields of science and mathematics and industrial arts. However, while teachers will continue to go overseas on individual assignments, increasing emphasis is being placed on the "project approach" to Aid, where Canada provides not only the services of educational advisers but, equally important, also trains counterparts who will replace the Canadians at the conclusion of their assignments. In addition Canada provides, when and where desirable, capital assistance in the form of constructing and equipping schools.

The following are examples of this project approach:

(i) *The University of the West Indies Programme*

involves the supply of Canadian professors, the provision of awards for study in Canada to senior UWI teaching staff members, and undergraduate training awards, tenable at the UWI, for students from the smaller territories. Capital assistance is also provided for the UWI building programme. The Canadian programme for the UWI as a whole extends over five years and is financed by a $5 million Canadian grant. It is expected that about one-third of the funds will be used for capital assistance, one-third for scholarships in Canada and at the UWI, and one-third for the provision of Canadian professors

(ii) *The Thailand Comprehensive School Project*

under which the University of Alberta provides expertise in terms of educationists currently serving in Thailand, while, at the same time, counterpart Thai students are studying in Canada. This project also includes the provision of equipment.

(iii) *The Accra Trades Training Centre*

where Canada not only supplied the advisers but also aided in the building and equipping of the Centre. The present Principal of the Accra Trades Training Centre is a Canadian and it is expected that not only he but also the other Canadians currently serving there will be replaced by Ghanaian students presently studying in Canada.

23. The Canadians serving overseas under Canada's Educational Assistance Programmes are all university graduates and professionally trained with a number of years of experience in the teaching profession. Normally they serve overseas for a minimum period of two years which may be extended to a maximum of five years by mutual agreement.

24. The Canadian view is that educational assistance to developing countries can benefit Canada as much as the recipient country and that Canadian educationists who have served overseas have come back to Canada better teachers, with a greater degree of understanding of other nations, their hopes and aspirations.

Australia

25. Australia provides educational assistance on a bilateral basis to Commonwealth countries in Africa, South and South-East Asia, the Pacific region and the West Indies under a number of programmes. These programmes are the Colombo Plan, the scheme of Commonwealth Co-operation in Education, the Special Commonwealth African Assistance Plan and the Australian South Pacific Technical Assistance Programme. With the exception of Commonwealth Co-operation in Education, these programmes offer technical assistance in a wide variety of fields which may include education if the Government requesting assistance give priority to that field.

26. The oldest of these programmes, the Colombo Plan, was inaugurated in 1950, and is concerned with the provision of economic aid and technical assistance to countries of South and South-East Asia, including the Commonwealth countries in that area.

27. With the establishment in 1959 of the scheme of *Commonwealth Co-operation in Education,* which includes Australian participation in the Commonwealth Scholarship and Fellowship Plan, Australia began to provide educational assistance to Commonwealth countries outside the Colombo Plan area. Since resources under this programme are fairly limited, however, it was not possible to offer educational assistance to all developing Commonwealth countries not covered by the Colombo Plan. A selection has therefore been made each year of the countries invited to nominate candidates for training or experience in Australia. At first these were countries in Africa and the Pacific, but since 1966 awards have also been available to some countries in the West Indies area.

28. Australian educational assistance under the above programmes takes two main forms, firstly, awards to provide training in teaching and other aspects of education; and secondly, the supply of Australian educational personnel to staff key posts in developing countries.

Awards to provide training

29. It is the Australian view that basic teacher-training is best conducted in the home environment. Where insufficient facilities are available in the nominating countries concerned, however, nominations are accepted for basic teacher-training especially secondary teacher-training consisting of a degree course followed by a postgraduate diploma in education. Such nominations are made principally under the Colombo Plan.

30. Training Awards under the scheme of Commonwealth Co-operation in Education are intended to provide mainly either specialist training for teachers who have completed their basic training, or training in administrative or supervisory aspects of education. Normally the length of course undertaken is one year or less. Similar courses may be made available under the Colombo Plan and other technical assistance programmes if requested.

31. When nominations are invited for Australian awards, no restriction is placed on the fields of education in which a nominating country may request training and places are not reserved in advance for particular courses in universities and colleges.

The fields chosen therefore reflect the interests of nominating countries and the aspects of Australian education which they regard as most relevant to their own needs.

32. In recent years a high proportion of the awards made under Commonwealth Co-operation in Education have been for the special courses organised under the scheme. Apart from these special courses, major fields of interest have been the teaching of trade subjects, domestic science, music, infant teaching methods, agriculture and teaching English as a second language. A number of nominations have been linked with the assignments carried out under the scheme by key Australian educational personnel. Requests for training specifically in the teaching of mathematics have not been common, although the primary methods course mentioned below has proved very popular.

33. To provide specifically for the interest shown by nominating countries in particular fields of study, a number of special courses have from time to time been arranged in Australia, varying in length from three months to one academic year. These courses have included the teaching of English as a second language, school broadcasting, teacher-training methods (Certificate in Education for Overseas Teachers), primary teaching methods, (including mathematics), school inspection and school administration.

34. Visitors Awards enable educationists of experience and standing in the practice and administration of education in their home countries to make short study visits to Australia. Not only do these awards widen the experience of the visitors, but especially in the case of the more senior officers, they also establish liaison between education authorities in Australia and those in developing Commonwealth countries and thus facilitate the most effective kinds of educational co-operation.

Supply of Educational Personnel

35. In general, Australia does not provide school teachers for overseas service, partly because of the prevailing teacher shortage, both in Australia and in the Territory of Papua and New Guinea, and partly because of the desire to apply Australian resources to projects having greater long-term effect.

36. It is the Australian policy to provide the kind of educationist who will start a new course, establish an educational institution, train local teachers, or, through supervisory or inspectorial duties, improve the quality of local teaching over a wide area. Numbers of such experts in different fields have been provided. They include technical education, curriculum development and correspondence education as well as established fields of teacher training.

A Project in Mid-West Nigeria

37. An example of the kind of project Australia is interested in supporting is that undertaken by three Australian educationists in Mid-West Nigeria from late 1964 to the end of October, 1966. During this time these educationists collaborated as substantive officers of the Ministry of Education in developing a system of education designed to meet the special needs of the people in this new political region. From May to September 1966, they were joined by two Australian infant method

specialists who, under the supervision of the senior member of the Australian team, introduced a course in infant teaching method.

Papua and New Guinea

38. A significant proportion of Australia's contribution towards programmes of educational co-operation is being directed towards the territory of Papua and New Guinea. Mr. J.W. Humphreys, of the Department of Education in the Territory attended the Conference as a member of the Australian delegation. With Professor Dienes, formerly of the University of Adelaide and now of the University of Sherbrooke in Canada as consultant and adviser, a primary school course in mathematics covering the first 3 years and based on Professor Dienes' work in South Australia has been introduced in the Territory Area for the period 1964 to 1968 and it is expected that the programme for the final three years of the primary course will have been completed before 1972.

The African Mathematics Programme of the Education Development Center

39. Among the programmes in Africa for the reform of the teaching of mathematics one of the best-known is that introduced by the Education Development Center. This organization was represented at the Trinidad Conference by Dr. Grace Williams, of the University of Lagos in Nigeria.

40. Education Development Center, Inc. (EDC), formerly Education Services Inc. (ESI), is a non-profit organization concerned with research and development in curriculum reform at all levels of education, from kindergarten to university level. The Ford Foundation, and other United States charitable bodies, as well as the National Science Foundation, have provided funds for its work. The overseas operations of EDC in Africa, India, Afghanistan and South America, have been supported by funds from U.S.A.I.D. and from the Ford Foundation and other philanthropic foundations.

41. In 1961 a Conference of African, British and American educators made firm proposals that EDC should initiate curricular reform programmes for Africa. In particular programmes in mathematics, science, social studies and teacher training were recommended. EDC agreed, initially, to try to act upon a programme for the reform of the teaching of mathematics from Standard I up to School Certificate. It was felt that the work of curriculum reform in mathematics in the U.S.A. and in Britain was sufficiently advanced to make possible a positive contribution to African education.

42. Under the direction of Professor W.T. Martin of the Massachusetts Institute of Technology, the programme was initiated and, after six months of activity including conferences in Accra and Ibadan with African mathematicians, a mathematics workshop was held at Entebbe, Uganda, in July and August 1962. A plan for the African Mathematics Programme was drawn up and the pattern of work has been followed since. There were fifty-four participants representing 13 countries, with 24 participants coming from eleven African countries. It was decided to produce materials using the new approach to teaching mathematics in four areas: text materials for the primary school, for the secondary school, text materials for

use in African teacher training colleges, and also tests and examinations based on these materials.

43. Since the programme started a total of 61 volumes of textual materials have been published. In these five years some 70 Africans, 20 expatriates, serving in Africa, 50 Americans and 5 British have helped to write the mathematical texts. In this EDC programme we see an example of an element of co-operation which might well be extended in future planning.

Examinations

44. In 1967, the West African Examinations Council set a School Certificate Examination on the work of the Entebbe Mathematics Series and the Cambridge Overseas Examination Syndicate has promised that when there is a demand a suitable examination will also be offered in East and Central Africa. Similar examination facilities are also available for the School Mathematics Project (Britain) and the Ghana Joint Schools Project.

Teacher Training

45. It became clear that it was necessary to introduce teachers to the new pedagogical approaches written into the new texts. This has been done through Institutes lasting from ten days to three weeks and staffed by visiting American mathematics educators and by their African counterparts who have participated in the programme. They are organized by the Ministry of Education in the participating country.

46. In addition to the in-country institutes described above a two-year institute, known as the ABC Institute, was started during July 1966, at the University College, Nairobi. This institute concerned itself with the mathematical education of the senior mathematics tutors of the participating countries. By the summer of 1969, these countries should have a small cadre of people knowledgeable about modern mathematics, capable of making adaptations of the materials to suit local conditions, and ready with suitable material to give help within their own countries to their fellow tutors in teachers' colleges, who in turn will give suitable training to their students and to teachers in the schools.

Co-operation within the Commonwealth

47. The large-scale EDC programme for improving and enlarging mathematical education in so many African territories pointed the way to the fuller development which a policy of co-operation between members of the Commonwealth could bring about. In general discussion, delegates at the Conference urged that Commonwealth countries should establish better communications with one another with regard to their experiments and achievements in the reform of mathematics teaching in schools, exchanging information about their schemes, the evaluation of experimental teaching, and the books and other materials that had been found useful. Exchange of personnel was suggested as a means of securing the cross-fertilization of ideas. Professional associations of mathematics teachers could well be of great service in fostering these means of communication, particularly if reciprocal affiliation could

be arranged between the various national associations. The help of the Commonwealth Foundation should be sought in facilitating these inter-association links. There was seen to be a need also for co-operation on a scale which would be beyond the resources of teachers' associations, especially in arranging exchanges of personnel, exhibitions of publications, films, video tapes etc. It was hoped that the Commonwealth Foundation, with its inter-Commonwealth structure, would be able to provide the channel through which such active collaboration could operate.

Co-operation in Aid within the Commonwealth

48. Some Commonwealth countries had already initiated new mathematical schemes, provided a large measure of further training for teachers and equipped many schools with the necessary apparatus and materials. These fortunate countries, in addition to the aid they now give in independent schemes to those less fortunate, could co-operate in the planning and establishment of similar enterprises in those countries which have too few resources to carry out, unaided, new programmes adapted to their local needs, as recommended by the Conference. If there could be a fusion of the contributions which could be most aptly made by several countries, there would be a unity of effort which would greatly benefit the receiving country. Not only would the aided country benefit by the coherence of planning and the greater strength of the united team of helpers; the contributing countries would be sharing ideas and experiments and thus enlarging the possible scope of their own programmes.

Bibliography of Book Exhibition

This bibliography was prepared during the Conference and the reader will understand that in the short time available it was not possible to prepare it in full library style. However, it was agreed that it would be helpful to teachers to have even minimal information about this collection of books which is now housed in the library of the University of the West Indies, St. Augustine, Trinidad, W.I.

We must apologise for any omission or other errors which may have occurred during the preparation of the bibliography.

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Association of Teachers of Mathematics, Vine Street Chambers, Nelson, Lancashire, England.

Association for Supervision and Curriculum Development, 1201, 16th Street, N.W., Washington D.C., 20036., U.S.A.

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Canadian Association of Mathematics Teachers, c/o. Canadian Teachers' Federation Office, 320, Queen Street, Ottawa, 4, Ont.

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School Mathematics Project, c/o Westfield College, Kidderpore Avenue, Hampstead, London, N.W.3. England.

School Mathematics Study Group, Cedar Hall, Stanford University, Stanford, California 93405, U.S.A.

Science Research Associates, 250 Erie Street, Chicago, Illinois 60611, U.S.A.

Scottish Mathematics Group, The Director, Rector, Hutchesons Boy's Grammar School, Beaton Road, Glasgow, Scotland.

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Printing Section
Marlborough House
London, S.W.1

S.B.N. 85092 000. 0.

ISBN 978-1-84859-168-4

9 781848 591684